Expansion of the integral $\int x^{f-1}dx(\log x)^{\frac{m}{n}}$ having extended the integral from the value $x=0$ to $x=1$ *

Leonhard Euler

Theorem 1

§1 *If n denotes a positive integer and the integral*

$$\int x^{f-1}dx(1-x^g)^n$$

is extended from the value $x=0$ to $x=1$, the value of the integral will be

$$=\frac{g^n}{f}\cdot\frac{1\cdot 2\cdot 3\cdots n}{(f+g)(f+2g)(f+3g)\cdots(f+ng)}.$$

Proof

It is known that the integral $\int x^{f-1}dx(1-x^g)^m$ in general can be reduced to this one $\int x^{f-1}dx(1-x^g)^{m-1}$, since it is possible to define constant quantities A and B in such a way that it is

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$$\int x^{f-1}dx(1-x^g)^m = A\int x^{f-1}dx(1-x^g)^{m-1} + Bx^f(1-x^g)^m;$$

for, by differentiation this equation results

$$\begin{aligned} &x^{f-1}dx(1-x^g)^m \\ &= Ax^{f-1}dx(1-x^g)^{m-1} + Bfx^{f-1}dx(1-x^g)^m - Bmgx^{f+g-1}dx(1-x^g)^{m-1}, \end{aligned}$$

which divided by $x^{f-1}dx(1-x^g)^{m-1}$ gives

$$1-x^g = A + Bf(1-x^g) - Bmgx^g$$

or

$$1-x^g = A - Bmg + B(f+mg)(1-x^g);$$

in order for this equation to hold, it is necessary that it is

$$1 = B(f+mg) \quad \text{and} \quad A = Bmg,$$

whence we conclude

$$B = \frac{1}{f+mg} \quad \text{and} \quad A = \frac{mg}{f+mg}.$$

Therefore, we will have the following general reduction

$$\int x^{f-1}dx(1-x^g)^m = \frac{mg}{f+mg}\int x^{f-1}dx(1-x^g)^{m-1} + \frac{1}{f+mg}x^f(1-x^g)^m;$$

because it vanishes for $x=0$, if $f>0$, of course, the addition of a constant is not necessary. Hence having extended both integrals to $x=1$ the last absolute part vanishes by itself and for the case $x=1$ it will be

$$\int x^{f-1}dx(1-x^g)^m = \frac{mg}{f+mg}\int x^{f-1}dx(1-x^g)^{m-1}.$$

Since for $m=1$ it is

$$\int x^{f-1}dx(1-x^g)^0 = \frac{1}{f}x^f = \frac{1}{f},$$

having put $x=1$, we obtain the following values for the same case $x=1$

$$\int x^{f-1}dx(1-x^g)^1 = \frac{g}{f} \cdot \frac{1}{f+g},$$
$$\int x^{f-1}dx(1-x^g)^2 = \frac{g^2}{f} \cdot \frac{1}{f+g} \cdot \frac{2}{f+2g},$$
$$\int x^{f-1}dx(1-x^g)^3 = \frac{g^3}{f} \cdot \frac{1}{f+g} \cdot \frac{2}{f+2g} \cdot \frac{3}{f+3g}$$

and hence we conclude that for any positive integer n it will be

$$\int x^{f-1}dx(1-x^g)^n = \frac{g^n}{f} \cdot \frac{1}{f+g} \cdot \frac{2}{f+2g} \cdot \frac{3}{f+3g} \cdots \frac{n}{f+ng},$$

if only the numbers f and g are positive.

COROLLARY 1

§2 Hence vice versa the value of a product of this kind, formed from an arbitrary amount of factors, can be expressed by an integral so that it is

$$\frac{1 \cdot 2 \cdot 3 \cdots n}{(f+g)(f+2g)(f+3g)\cdots(f+ng)} = \frac{f}{g^n}\int x^{f-1}dx(1-x^g)^n$$

having extended this integral from the value $x=0$ to $x=1$.

COROLLARY 2

§3 Therefore, if one considers a progression of this kind

$$\frac{1}{f+g}, \quad \frac{1 \cdot 2}{(f+g)(f+2g)}, \quad \frac{1 \cdot 2 \cdot 3}{(f+g)(f+2g)(f+3g)}, \quad \frac{1 \cdot 2 \cdot 3 \cdot 4}{(f+g)(f+2g)(f+3g)(f+4g)} \quad \text{etc.},$$

its general term corresponding to the indefinite index n, is conveniently represented by this integral $\frac{f}{g^n}\int x^{f-1}dx(1-x^g)^n$; and using this formula the progression and its terms corresponding to fractional indices can be exhibited.

COROLLARY 3

§4 If we write $n-1$ instead of n, we will have

$$\frac{1 \cdot 2 \cdot 3 \cdots (n-1)}{(f+g)(f+2g)(f+3g)\cdots(f+(n-1)g)} = \frac{f}{g^{n-1}} \int x^{f-1}dx(1-x^g)^{n-1},$$

which equation multiplied by $\frac{n}{f+ng}$ yields

$$\frac{1 \cdot 2 \cdot 3 \cdots n}{(f+g)(f+2g)(f+3g)\cdots(f+ng)} = \frac{f \cdot ng}{g^n(f+ng)} \int x^{f-1}dx(1-x^g)^{n-1}.$$

SCHOLIUM 1

It would have been possible to derive this last formula immediately from the preceding one, since we just proved that it is

$$\int x^{f-1}dx(1-x^g)^n = \frac{ng}{f+ng} \int x^{f-1}dx(1-x^g)^{n-1},$$

if both integrals are extended from the value $x=0$ to $x=1$; this is to be kept in mind for all the integrals in everything that follows. Furthermore, it is to be noted that the quantities f and g are positive, which condition was used in the proof, of course. Concerning the number n, if it denotes the index of a certain term of the progression (§3), that index can also be negative, because all terms, also the ones corresponding to negative indices, of the progression are considered to be exhibited by the given integral formula. Nevertheless, it is to be noted that this reduction

$$\int x^{f-1}dx(1-x^g)^m = \frac{mg}{f+mg} \int x^{f-1}dx(1-x^g)^{m-1}$$

is only true, if $m>0$, because otherwise the algebraic part $\frac{1}{f+mg}x^f(1-x^g)^m$ would non vanish for $x=1$.

SCHOLIUM 2

§6 I already studied series of this kind, which can be called transcendental, because the terms corresponding to fractional indices, are transcendental quantities, in COMMENT. ACAD. SC. PETROP., BOOK 5 in more detail[1]; therefore, I will not investigate those progressions here again but focus on the remarkable

[1]Euler refers to his paper "De progressionibus transcendentibus seu quarum termini generales algebraice dari nequeunt." This is paper E19 in the Eneström-Index

comparisons of the integral formulas, which are derived from it. After I had shown that the value of the indefinite product $1 \cdot 2 \cdot 3 \cdots n$ is expressed by the integral formula $\int dx \left(\log \frac{1}{x}\right)^n$ extended from $x = 0$ to $x = 1$, which, if n is a positive integer, is manifest by direct integration, I examined the cases, in which a fractional number is taken for n; in these cases it is indeed not obvious at all, to which kind of transcendental quantities these terms are to be referred. But by a singular artifice I reduced the same terms to better-known quadratures; therefore, this seems to be worth one's while to consider it with all eagerness.

PROBLEM 1

§7 *Since it was demonstrated that it is*

$$\frac{1 \cdot 2 \cdot 3 \cdot n}{(f+g)(f+2g)(f+3g)\cdots(f+ng)} = \frac{f}{g^n} \int x^{f-1} dx (1-x^g)^n$$

having extended the integral from $x = 0$ to $x = 1$, to assign the value of the same product in the case $g = 0$ by means of an integral.

SOLUTION

Having put $g = 0$ in the integral the term $(1-x^g)^n$ vanishes, but at the same time also the denominator g^n vanishes, whence the question reduces to the task that the value of the fraction $\frac{(1-x^g)^n}{g^n}$ is defined in the case $g = 0$, in which so the numerator as the denominator vanishes. Hence let us consider g as an infinitely small quantity, and because it is $x^g = e^{g \log x}$, it will be $x^g = 1 + g \log x$ and hence $(1-x^g)^n = g^n(-\log x)^n = g^n \left(\log \frac{1}{x}\right)^n$; hence our integral becomes $f \int x^{f-1} dx \left(\log \frac{1}{x}\right)^n$ for this case so that one now has this expression

$$\frac{1 \cdot 2 \cdot 3 \cdots n}{f^n} = f \int x^{f-1} dx \left(\log \frac{1}{x}\right)^n$$

or

$$1 \cdot 2 \cdot 3 \cdots n = f^{n+1} \int x^{f-1} dx \left(\log \frac{1}{x}\right)^n.$$

COROLLARY 1

§8 If n is a positive integer, the integration of the integral $\int x^{f-1}dx\big(\log\frac{1}{x}\big)^n$ succeeds and having extended it from $x=0$ to $x=1$ indeed the product we found to be equal to it results. But if fractional numbers are taken for n, the same formula can be applied to interpolate this hypergeometric progression

$$1, \quad 1\cdot 2, \quad 1\cdot 2\cdot 3, \quad 1\cdot 2\cdot 3\cdot 4, \quad 1\cdot 2\cdot 3\cdot 4\cdot 5 \quad \text{etc.}$$

or

$$1, \quad 2, \quad 6, \quad 24, \quad 24, \quad 120, \quad 720, \quad 5040 \quad \text{etc.}$$

COROLLARY 2

§9 If the expression just found is divided by the principal one, a product whose factors proceed in an arithmetic progression will emerge, namely

$$(f+g)(f+2g)(f+3g)\cdots(f+ng) = f^n g^n \frac{\int x^{f-1}dx\,\big(\log\frac{1}{x}\big)^n}{\int x^{f-1}dx\,(1-x^g)^n},$$

whose values can also be assigned using the integral, if n is a fractional number.

COROLLARY 3

§10 Because it is

$$\int x^{f-1}dx(1-x^g)^n = \frac{ng}{f+ng}\int x^{f-1}dx(1-x^g)^{n-1},$$

in like manner it will be for the case $g=0$

$$\int x^{f-1}dx\left(\log\frac{1}{x}\right)^n = \frac{n}{f}\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{n-1}$$

and hence by those other integrals

$$1\cdot 2\cdot 3\cdots n = nf^n\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{n-1}$$

and

$$(f+g)(f+2g)\cdots(f+ng) = f^{n-1}g^{n-1}(f+ng)\frac{\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{n-1}}{\int x^{f-1}dx(1-x^g)^{n-1}}.$$

SCHOLIUM

§11 Because we found that

$$1\cdot 2\cdot 3\cdots n = f^{n+1}\int x^{f-1}dx\left(\log\frac{1}{x}\right)^n,$$

it is plain that this integral does not depend on the value of the quantity f, what is also easily seen by putting $x^f = y$, whence we first find

$$fx^{f-1}dx = dy \quad \text{and} \quad \log\frac{1}{x} = -\log x = -\frac{1}{f}\log y = \frac{1}{f}\log\frac{1}{y}$$

and therefore

$$f^n\left(\log\frac{1}{x}\right)^n = \left(\log\frac{1}{y}\right)^n,$$

so that it is

$$1\cdot 2\cdot 3\cdots n = \int dy\left(\log\frac{1}{y}\right)^n,$$

which expression results from the first by putting $f = 1$. Therefore, for an interpolation of this kind the whole task is reduced to definition of the values of the integral $\int dx\left(\log\frac{1}{x}\right)^n$ for the cases, in which the exponent n is a fractional number. For example, if it is $n = \frac{1}{2}$, one has to assign the value of the formula $\int dx\sqrt{\log\frac{1}{x}}$, which value I already once showed to be $= \frac{1}{2}\sqrt{\pi}$, while π denotes the circumference of the circle, whose diameter is $= 1$; but for other fractional numbers I taught how to reduce its value to quadratures of algebraic curves of higher order. Because this reduction is by no means obvious and is only valid, if the integration of the formula $\int dx\left(\log\frac{1}{x}\right)^n$ is extended from the value $x = 0$ to $x = 1$, it is seems to be worth one's attention. But even though I already treated this subject once[2], I nevertheless, because I was led to the results in a rather non straight-forward way, decided take on this subject here again and explain everything in more detail.

[2]Euler considered this expression also in E19, mentioned already in the footnote above.

THEOREM 2

§12 *If the integrals are extended from the value $x = 0$ to $x = 1$ and n denotes a positive integer, it will be*

$$\frac{1\cdot 2\cdot 3\cdots n}{(n+1)(n+2)(n+2)\cdots 2n} = \frac{1}{2}ng\int x^{f+ng-1}dx(1-x^g)^{n-1}\cdot\frac{\int x^{f-1}dx(1-x^g)^{n-1}}{\int x^{f-1}dx(1-x^g)^{2n-1}},$$

no matter which positive numbers are taken for f and g.

PROOF

Because above (§4) we showed that it is

$$\frac{1\cdot 2\cdot 3\cdots n}{(f+g)(f+2g)\cdots(f+ng)} = \frac{f\cdot ng}{g^n(f+ng)}\int x^{f-1}dx(1-x^g)^{n-1},$$

if we write $2n$ instead of n, we will have

$$\frac{1\cdot 2\cdot 3\cdots 2n}{(f+g)(f+2g)\cdots(f+2ng)} = \frac{f\cdot 2ng}{g^{2n}(f+2ng)}\int x^{f-1}dx(1-x^g)^{2n-1}.$$

Now divide the first equation by the second and this third one will result

$$\frac{(f+(n+1)g)(f+(n+2)g)\cdots(f+2ng)}{(n+1)(n+2)\cdots 2n} = \frac{g^n(f+2ng)}{2(f+ng)}\cdot\frac{\int x^{f-1}dx(1-x^g)^{n-1}}{\int x^{f-1}dx(1-x^g)^{2n-1}}.$$

But if one writes $f+ng$ instead of f in the first equation, this fourth equation will result

$$\frac{1\cdot 2\cdot 3\cdots n}{(f+(n+1)g)(f+(n+2)g)\cdots(f+2ng)} = \frac{(f+ng)ng}{g^n(f+2ng)}\int x^{f+ng-1}dx(1-x^g)^{n-1}.$$

Multiply this fourth equation by the third one and one will find the equation to be demonstrated, namely

$$\frac{1\cdot 2\cdot 3\cdots n}{(n+1)(n+2)(n+3)\cdots 2n} = \frac{1}{2}ng\int x^{f+ng-1}dx(1-x^g)^{n-1}\cdot\frac{\int x^{f-1}dx(1-x^g)^{n-1}}{\int x^{f-1}dx(1-x^g)^{2n-1}}.$$

COROLLARY 1

§13 If one sets $f = n$ and $g = 1$ in the first equation, the same product will result, of course

$$\frac{1 \cdot 2 \cdot 3 \cdots n}{(n+1)(n+2)\cdots 2n} = \frac{1}{2}n \int x^{n-1}dx(1-x^g)^{n-1};$$

having compared this equation to the one mentioned we obtain

$$\frac{\int x^{n-1}dx(1-x)^{n-1}}{g\int x^{f+ng-1}dx(1-x^g)^{n-1}} = \frac{\int x^{f-1}dx(1-x^g)^{n-1}}{\int x^{f-1}dx(1-x^g)^{2n-1}}.$$

COROLLARY 2

§14 If we write x^g instead of x in that equation, it will be

$$\frac{1 \cdot 2 \cdot 3 \cdots n}{(n+1)(n+2)\cdots 2n} = \frac{1}{2}ng \int x^{ng-1}dx(1-x^g)^{n-1},$$

so that we find this comparison of the following integral formulas

$$\int x^{ng-1}dx(1-x^g)^{n-1} = \int x^{f+ng-1}dx(1-x^g)^{n-1} \cdot \frac{\int x^{f-1}dx(1-x^g)^{n-1}}{\int x^{f-1}dx(1-x^g)^{2n-1}}.$$

COROLLARY 3

§15 If we put $g = 0$ in the equation of the theorem, because of $(1-x^g)^m = g^m \left(\log \frac{1}{x}\right)^m$ the powers of g will cancel each other and this equation will result

$$\frac{1 \cdot 2 \cdot 3 \cdots n}{(n+1)(n+2)\cdots 2n} = \frac{1}{2}n \int x^{f-1}dx \left(\log \frac{1}{x}\right)^{n-1} \cdot \frac{\int x^{f-1}dx \left(\log \frac{1}{x}\right)^{n-1}}{\int x^{f-1}dx \left(\log \frac{1}{x}\right)^{2n-1}},$$

whence we conclude

$$\frac{\left(\int x^{f-1}dx \left(\log \frac{1}{x}\right)^{n-1}\right)^2}{\int x^{f-1}dx \left(\log \frac{1}{x}\right)^{2n-1}} = g \int x^{ng-1}dx(1-x^g)^{n-1}$$

or because of

$$\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{n-1}=\frac{f}{n}\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{n}$$

this one

$$\frac{2f}{n}\cdot\frac{\left(\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{n}\right)^{2}}{\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{2n}}=g\int x^{ng-1}dx(1-x^{g})^{n-1}.$$

COROLLARY 4

§16 Let us put $f=1$, $g=2$ and $n=\frac{m}{2}$ here so that m is a positive integer, and because of

$$\int dx\left(\log\frac{1}{x}\right)^{m}=1\cdot 2\cdot 3\cdots m$$

it will be

$$\frac{4}{m}\cdot\frac{\left(\int dx\left(\log\frac{1}{x}\right)^{\frac{m}{2}}\right)^{2}}{1\cdot 2\cdot 3\cdots m}=2\int x^{m-1}dx(1-x^{2})^{\frac{m}{2}-1}$$

and hence

$$\int dx\left(\log\frac{1}{x}\right)^{\frac{m}{2}}=\sqrt{1\cdot 2\cdot 3\cdots m\cdot\frac{m}{2}\int x^{m-1}dx(1-x^{2})^{\frac{m}{2}-1}}$$

and by taking $m=1$, because of

$$\int\frac{dx}{\sqrt{1-xx}}=\frac{\pi}{2},$$

one will have

$$\int dx\sqrt{\log\frac{1}{x}}=\sqrt{\frac{1}{2}\int\frac{dx}{\sqrt{1-xx}}}=\frac{1}{2}\sqrt{\pi}.$$

SCHOLIUM

§17 So behold this succinct proof of the theorem I once propounded and stating that it is $\int dx\sqrt{\log\frac{1}{x}}=\frac{1}{2}\sqrt{\pi}$, and note that I did not use an argument involving interpolations, which I had used back then. Here it was of course deduced from this theorem I found here and which states that it is

$$\frac{\left(\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{n-1}\right)^2}{\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{2n-1}} = g\int x^{ng-1}dx(1-x^g)^{n-1}.$$

But the principal theorem, whence this one is deduced, reads as follows

$$g\frac{\int x^{f-1}dx(1-x^g)^{n-1}\cdot\int x^{f+ng-1}dx(1-x^g)^{n-1}}{\int x^{f-1}dx(1-x^g)^{2n-1}} = \int x^{n-1}dx(1-x)^{n-1};$$

for, each side, if it is actually calculated by an integration extended from $x=0$ to $x=1$, is equal to this product

$$\frac{1\cdot 2\cdot 3\cdots(n-1)}{(n+1)(n+2)\cdots(2n-1)}.$$

But if we want to give the one side a more general form involving a further-extending class of integrals, we can state the theorem in such a way that it is

$$g\frac{\int x^{f-1}dx(1-x^g)^{n-1}\cdot\int x^{f+ng-1}dx(1-x^g)^{n-1}}{\int x^{f-1}dx(1-x^g)^{2n-1}} = k\int x^{nk-1}dx(1-x^k)^{n-1};$$

and if here one takes $g=0$, it is

$$\frac{\left(\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{n-1}\right)^2}{\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{2n-1}} = k\int x^{nk-1}dx(1-x^k)^{n-1}.$$

Therefore, it is especially to be noted that that equality holds, no matter which numbers are taken for f and g; in the case $f=g$ this is indeed clear, because it is

$$\int x^{g-1}dx(1-x^g)^{n-1} = \frac{1-(1-x^g)^n}{ng} = \frac{1}{ng};$$

for, it will be

$$2g\int x^{ng+g-1}dx(1-x^g)^{n-1} = k\int x^{nk-1}dx(1-x^k)^{n-1},$$

and because

$$\int x^{ng+g-1}dx(1-x^g)^{n-1} = \frac{1}{2}\int x^{ng-1}dx(1-x^g)^{n-1},$$

the equality is perspicuous, because k can be taken arbitrarily. But in the same way we got to this theorem, it is possible to get to other similar ones.

THEOREM 3

§18 *If the following integrals are extended from the value $x = 0$ to $x = 1$ and n denotes any positive integer, it will be*

$$\frac{1\cdot 2\cdot 3\cdots n}{(2n+1)(2n+2)\cdots 3n} = \frac{2}{3}ng\int x^{f+2ng-1}dx(1-x^g)^{n-1}\cdot\frac{\int x^{f-1}dx(1-x^g)^{2n-1}}{\int x^{f-1}dx(1-x^g)^{3n-1}},$$

no matter which positive numbers are taken for f and g.

PROOF

In the preceding theorem we already saw that

$$\frac{1\cdot 2\cdot 3\cdots 2n}{(f+g)(f+2g)\cdots(f+2ng)} = \frac{f\cdot 2ng}{g^{2n}(f+2ng)}\int x^{f-1}dx(1-x^g)^{2n-1};$$

if in like manner we write $3n$ instead of n in the principal formula, we will have

$$\frac{1\cdot 2\cdot 3\cdots 3n}{(f+g)(f+2g)\cdots(f+3ng)} = \frac{f\cdot 3ng}{g^{3n}(f+3ng)}\int x^{f-1}dx(1-x^g)^{3n-1};$$

hence dividing this equation by the first one is led to

$$\frac{(f+(2n+1)g)(f+(2n+2)g)\cdots(f+3ng)}{(2n+1)(2n+2)\cdots 3n} = \frac{2g^n(f+3ng)}{3(f+2ng)}\cdot\frac{\int x^{f-1}dx(1-x^g)^{2n-1}}{\int x^{f-1}dx(1-x^g)^{3n-1}}.$$

But if we write $f+2gn$ instead of f in the principal equation (§4), we obtain this equation

$$\frac{1\cdot 2\cdot 3\cdots n}{(f+(2n+1)g)(f+(2n+2)g)\cdots(f+3ng)}=\frac{(f+2ng)ng}{g^n(f+3ng)}\int x^{f+2ng-1}dx(1-x^g)^{n-1}.$$

Now multiply this equation by the preceding and the equation to be proved will result

$$\frac{1\cdot 2\cdot 3\cdots n}{(2n+1)(2n+2)\cdot 3n}=\frac{2}{3}ng\int x^{f+2ng-1}dx(1-x^g)^{n-1}\cdot\frac{\int x^{f-1}dx(1-x^g)^{2n-1}}{\int x^{f-1}dx(1-x^g)^{3n-1}}.$$

COROLLARY 1

§19 We obtain the same value from the principal equation by putting $f=2n$ and $g=1$ so that

$$\frac{1\cdot 2\cdot 3\cdots n}{(2n+1)(2n+2)\cdots 3n}=\frac{2}{3}n\int x^{2n-1}dx(1-x)^{n-1},$$

which integral formula, by writing x^k instead of x, is transformed into this one

$$\frac{2}{3}nk\int x^{2nk-1}dx(1-x^k)^{n-1},$$

so that

$$g\int x^{f+2ng-1}dx(1-x^g)^{n-1}\cdot\frac{\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{2n-1}}{\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{3n-1}}=k\int x^{2nk-1}dx(1-x^k)^{n-1}.$$

COROLLARY 2

§20 If we set $g=0$ here, because of $1-x^g=g\log\frac{1}{x}$ we will have this equation

$$\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{n-1}\cdot\frac{\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{2n-1}}{\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{3n-1}}=k\int x^{2nk-1}dx(1-x^k)^{n-1};$$

because we had found before that it is

$$\frac{\left(\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{n-1}\right)^2}{\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{2n-1}} = k\int x^{nk-1}dx(1-x^k)^{n-1},$$

by multiplying both expressions by each other we will have this equation

$$\frac{\left(\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{n-1}\right)^2}{\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{3n-1}} = k^2\int x^{nk-1}dx(1-x^k)^{n-1}\cdot\int x^{2nk-1}dx(1-x^k)^{n-1}.$$

COROLLARY 3

§21 Without any restriction one can put $f=1$ here; for, then for $n=\frac{1}{3}$ and $k=3$ it will be

$$\frac{\left(\int dx\left(\log\frac{1}{x}\right)^{-\frac{2}{3}}\right)^3}{\int dx\left(\log\frac{1}{x}\right)^0} = 9\int dx(1-x^3)^{-\frac{2}{3}}\cdot\int xdx(1-x^3)^{-\frac{2}{3}}$$

and because of

$$\int dx\left(\log\frac{1}{x}\right)^{-\frac{2}{3}} = 3\int dx\left(\log\frac{1}{x}\right)^{\frac{1}{3}} \quad \text{and} \quad \int dx\left(\log\frac{1}{x}\right)^0 = 1$$

$$\left(\int dx\left(\log\frac{1}{x}\right)^{\frac{1}{3}}\right)^3 = \frac{1}{3}\int dx(1-x^3)^{-\frac{2}{3}}\cdot\int xdx(1-x^3)^{-\frac{2}{3}};$$

but then for $n=\frac{2}{3}$ and $k=3$ it will be

$$\frac{\left(\int dx\left(\log\frac{1}{x}\right)^{-\frac{1}{3}}\right)^3}{\int dx\log\frac{1}{x}} = 9\int xdx(1-x^3)^{-\frac{1}{3}}\cdot\int x^3dx(1-x^3)^{-\frac{1}{3}}$$

or

$$\left(\int dx\left(\log\frac{1}{x}\right)^{\frac{2}{3}}\right)^3 = \frac{4}{3}\int xdx(1-x^3)^{-\frac{1}{3}}\cdot\int x^3dx(1-x^3)^{-\frac{1}{3}}.$$

General Theorem

§22 *If the following integrals are extended from the value $x = 0$ to $x = 1$ and n denotes a positive integer, it will be*

$$\frac{1 \cdot 2 \cdot 3 \cdots n}{(\lambda n+1)(\lambda n+2)\cdots(\lambda+1)n} = \frac{\lambda}{\lambda+1} ng \int x^{f+\lambda ng-1}dx(1-x^g)^{n-1} \cdot \frac{\int x^{f-1}dx(1-x^g)^{\lambda n-1}}{\int x^{f-1}dx(1-x^g)^{(\lambda+1)n-1}},$$

no matter which positive numbers are taken for the letters f and g.

Proof

Because, as we showed above, it is

$$\frac{1 \cdot 2 \cdots n}{(f+g)(f+2g)\cdots(f+ng)} = \frac{f \cdot ng}{g^n(f+ng)} \int x^{f-1}dx(1-x^g)^{n-1},$$

if we write λn instead of n here at first, but then $(\lambda+1)n$ instead of n, we will obtain these two equations

$$\frac{1 \cdot 2 \cdots \lambda n}{(f+g)(f+2g)\cdots(f+\lambda ng)} = \frac{f \cdot \lambda ng}{g^{\lambda n}(f+\lambda ng)} \int x^{f-1}dx(1-x^g)^{\lambda n-1},$$

$$\frac{1 \cdot 2 \cdots (\lambda+1)n}{(f+g)(f+2g)\cdots(f+(\lambda+1)ng)} = \frac{f \cdot (\lambda+1)ng}{g^{(\lambda+1)n}(f+(\lambda+1)ng)} \int x^{f-1}dx(1-x^g)^{(\lambda+1)n-1};$$

dividing the first equation by this one gives

$$\frac{(f+\lambda ng+g)(f+\lambda ng+2g)\cdots(f+\lambda ng+ng)}{(\lambda n+1)(\lambda n+2)\cdots(\lambda n+n)} = g^n \frac{\lambda(f+\lambda ng+ng)}{(\lambda+1)(f+\lambda ng)} \cdot \frac{\int x^{f-1}dx(1-x^g)^{\lambda n-1}}{\int x^{f-1}dx(1-x^g)^{(\lambda+1)n-1}}.$$

But if we write $f+\lambda ng$ instead of f in the first equation, we will obtain

$$\frac{1 \cdot 2 \cdots n}{(f+\lambda ng+g)(f+\lambda ng+2g)\cdots(f+\lambda ng+ng)} = \frac{(f+\lambda ng)ng}{g^n(f+\lambda ng+ng)} \int x^{f+\lambda ng-1}dx(1-x^g)^{n-1},$$

which two equations multiplied by each other produce the equation to be demonstrated

$$\frac{1 \cdot 2 \cdots n}{(\lambda n+1)(\lambda n+2)\cdots(\lambda n+n)} = \frac{\lambda ng}{\lambda+1} \int x^{f+\lambda ng-1}dx(1-x^g)^{n-1} \cdot \frac{\int x^{f-1}dx(1-x^g)^{\lambda n-1}}{\int x^{f-1}dx(1-x^g)^{(\lambda+1)n-1}}.$$

COROLLARY 1

§23 If we put $f = \lambda n$ and $g = 1$ in the principal equation, we will also find

$$\frac{1 \cdot 2 \cdots n}{(\lambda n + 1)(\lambda n + 2) \cdots (\lambda n + n)} = \frac{\lambda n}{\lambda + 1} \int x^{\lambda n - 1} dx (1 - x)^{n-1},$$

which form, by writing x^k instead of x changes into this one

$$\frac{\lambda n k}{\lambda + 1} \int x^{\lambda n k - 1} dx (1 - x^k)^{n-1},$$

so that we have this very far-extending theorem

$$g \int x^{f + \lambda n g - 1} dx (1 - x^g)^{n-1} \cdot \frac{\int x^{f-1} dx (1 - x^g)^{\lambda n - 1}}{\int x^{f-1} dx (1 - x^g)^{\lambda n + n - 1}} = k \int x^{\lambda n k - 1} dx (1 - x^k)^{n-1}.$$

COROLLARY 2

§24 This theorem now holds, even if n is not an integer; let us even, because the number λ can be taken arbitrarily, write m instead of λn and we will find this theorem

$$\frac{\int x^{f-1} dx (1 - x^g)^{m-1}}{\int x^{f-1} dx (1 - x^g)^{m+n-1}} = \frac{k \int x^{mk-1} dx (1 - x^k)^{n-1}}{g \int x^{f+mg-1} dx (1 - x^g)^{n-1}}.$$

COROLLARY 3

§25 If we put $g = 0$, because of $1 - x^g = g \log \frac{1}{x}$ that theorem will take this form

$$\frac{\int x^{f-1} dx \left(\log \frac{1}{x}\right)^{m-1}}{\int x^{f-1} dx \left(\log \frac{1}{x}\right)^{m+n-1}} = \frac{k \int x^{mk-1} dx (1 - x^k)^{n-1}}{\int x^{f-1} dx \left(\log \frac{1}{x}\right)^{n-1}},$$

which is more conveniently represented as follows

$$\frac{\int x^{f-1} dx \left(\log \frac{1}{x}\right)^{n-1} \cdot \int x^{f-1} dx \left(\log \frac{1}{x}\right)^{m-1}}{\int x^{f-1} dx \left(\log \frac{1}{x}\right)^{m+n-1}} = k \int x^{mk-1} dx (1 - x^k)^{n-1};$$

here it is evident that the numbers m and n can be permuted.

SCHOLIUM

§26 So we found two ways, how many comparisons and relations of integrals formulas can be derived; the one way, found in § 24, contains integrals of this kind

$$\int x^{p-1}dx(1-x^g)^{q-1},$$

which I already treated some time ago in my observations on the integrals[3]

$$\int x^{p-1}dx(1-x^n)^{\frac{q}{n}-1},$$

extended from the value $x=0$ to $x=1$; there I showed at first that the letters p and q can be interchanged that it is

$$\int x^{p-1}dx(1-x^n)^{\frac{q}{n}-1}=\int x^{q-1}dx(1-x^n)^{\frac{p}{n}-1},$$

but then that

$$\int \frac{x^{p-1}dx}{(1-x^n)^{\frac{p}{n}}}=\frac{\pi}{n\sin\frac{p\pi}{n}};$$

But especially I demonstrated that it is

$$\int \frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}\cdot\int \frac{x^{p+q-1}dx}{\sqrt[n]{(1-x^n)^{n-r}}}=\int \frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-r}}}\cdot\int \frac{x^{p+r-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}};$$

the comparison found in § 24 is already contained in this equation, so that nothing new, I did not already explain, can be deduced from this. Therefore, here I mainly attempt to follow the other way explained in § 25; since without any restriction one can take $f=1$, our primary equation will be

$$\frac{\int dx\left(\log\frac{1}{x}\right)^{n-1}\cdot\int dx\left(\log\frac{1}{x}\right)^{m-1}}{\int dx\left(\log\frac{1}{x}\right)^{m+n-1}}=k\int x^{mk-1}dx(1-x^k)^{n-1},$$

by means of which the values of the integral formula $\int dx\left(\log\frac{1}{x}\right)^{\lambda}$, if λ is not an integer, can be reduced to quadratures of algebraic curves; since, if λ is an integer, the integrals can be solved explicitly, because it is

[3] Euler again refers to his paper "Observationes circa integralia formularum $\int x^{p-1}dx(1-x^n)^{\frac{q}{n}-1}$ posito post integrationem $x=1$". This is paper E321 in the Eneström-Index.

$$\int dx \left(\log \frac{1}{x}\right)^{\lambda} = 1 \cdot 2 \cdot 3 \cdots \lambda.$$

But the question of greatest importance concerns the cases, in which λ is a rational number. Therefore, I will define these here successively for some small denominators.

PROBLEM 2

§27 *While i denotes a positive integer, to define the value of the integral $\int dx \left(\log \frac{1}{x}\right)^{\frac{i}{2}}$, having extended the integration from $x = 0$ to $x = 1$.*

SOLUTION

Let us put $m = n$ in our general equation and it will be

$$\frac{\left(\int dx \left(\log \frac{1}{x}\right)^{n-1}\right)^2}{\int dx \left(\log \frac{1}{x}\right)^{2n-1}} = k \int x^{nk-1} dx (1 - x^k)^{n-1}.$$

Now let it be $n - 1 = \frac{i}{2}$ and because of $2n - 1 = i + 1$ it will be

$$\int dx \left(\log \frac{1}{x}\right)^{2n-1} = 1 \cdot 2 \cdot 3 \cdots (i+1);$$

now further take $k = 2$ that it is $nk - 1 = i + 1$, and it will be

$$\frac{\left(\int dx \sqrt{\left(\log \frac{1}{x}\right)^i}\right)^2}{1 \cdot 2 \cdot 3 \cdots (i+1)} = 2 \int x^{i+1} dx (1 - x^2)^{\frac{i}{2}}$$

and hence

$$\frac{\int dx \sqrt{\left(\log \frac{1}{x}\right)^i}}{\sqrt{1 \cdot 2 \cdot 3 \cdots (i+1)}} = \sqrt{2 \int x^{i+1} dx (1 - x^2)^{\frac{i}{2}}},$$

where it is evident that it is sufficient to take only odd numbers for i, because for the even ones the expansion is manifest immediately.

COROLLARY 1

§28 But all cases are easily reduced to $i = 1$ or even to $i = -1$; for, if $i + 1$ is not a negative number, the reduction we found holds. For this case it will therefore be

$$\int \frac{dx}{\sqrt{\log \frac{1}{x}}} = \sqrt{2 \int \frac{dx}{\sqrt{1-xx}}} = \sqrt{\pi},$$

because of $\int \frac{dx}{\sqrt{1-xx}} = \frac{\pi}{2}$.

COROLLARY 2

§29 But having covered these principal cases, because of

$$\int dx \left(\log \frac{1}{x}\right)^{n} = n \int dx \left(\log \frac{1}{x}\right)^{n-1},$$

we will have

$$\int dx \sqrt{\log \frac{1}{x}} = \frac{1}{2}\sqrt{\pi}, \quad \int dx \left(\log \frac{1}{x}\right)^{\frac{3}{2}} = \frac{1 \cdot 3}{2 \cdot 2}\sqrt{\pi},$$

and in general

$$\int dx \left(\log \frac{1}{x}\right)^{\frac{2n+1}{2}} = \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \cdot \frac{7}{2} \cdots \frac{2n+1}{2}\sqrt{\pi}.$$

PROBLEM 3

§30 *While i denotes a positive integer, to define the value of the integral* $\int dx \left(\log \frac{1}{x}\right)^{\frac{i}{3}-1}$, *having extended the integration from* $x = 0$ *to* $x = 1$.

SOLUTION

Let us start from the equation of the preceding problem

$$\frac{\left(\int dx \left(\log \frac{1}{x}\right)^{n-1}\right)^2}{\int dx \left(\log \frac{1}{x}\right)^{2n-1}} = k \int x^{nk-1} dx (1-x^k)^{n-1}$$

and let us put $m = 2n$ in the general formula so that one has

$$\frac{\int dx \left(\log \frac{1}{x}\right)^{n-1} \cdot \int dx \left(\log \frac{1}{x}\right)^{2n-1}}{\int dx \left(\log \frac{1}{x}\right)^{3n-1}} = k \int x^{2nk-1} dx (1-x^k)^{n-1},$$

and by multiplying these two equations we obtain

$$\frac{\left(\int dx \left(\log \frac{1}{x}\right)^{n-1}\right)^3}{\int dx \left(\log \frac{1}{x}\right)^{3n-1}} = kk \int x^{nk-1} dx (1-x^k)^{n-1} \cdot \int x^{2nk-1} dx (1-x^k)^{n-1}.$$

Now just put $n = \frac{i}{3}$ here that it is

$$\int dx \left(\log \frac{1}{x}\right)^{i-1} = 1 \cdot 2 \cdot 3 \cdots (i-1),$$

and take $k = 3$ and this equation will result

$$\frac{\left(\int dx \sqrt[3]{\left(\log \frac{1}{x}\right)^{i-3}}\right)^3}{1 \cdot 2 \cdot 3 \cdots (i-1)} = 9 \int x^{i-1} dx \sqrt[3]{(1-x^3)^{i-3}} \cdot \int x^{2i-1} dx \sqrt[3]{(1-x^3)^{i-3}},$$

whence we conclude

$$\frac{\int dx \sqrt[3]{\left(\log \frac{1}{x}\right)^{i-3}}}{\sqrt{1 \cdot 2 \cdot 3 \cdots (i-1)}} = \sqrt[3]{9 \int \frac{x^{i-1} dx}{\sqrt[3]{(1-x^3)^{3-i}}} \cdot \int \frac{x^{2i-1} dx}{\sqrt[3]{(1-x^3)^{3-i}}}}$$

COROLLARY 1

§31 Here two principal cases occur, on which all remaining ones depend, namely the cases $i = 1$ and $i = 2$; for these cases it is

$$\text{I.} \quad \int \frac{dx}{\sqrt[3]{\left(\log \frac{1}{x}\right)^2}} = \sqrt[3]{9 \int \frac{dx}{\sqrt[3]{(1-x^3)^2}} \cdot \int \frac{x dx}{\sqrt[3]{(1-x^3)^2}}},$$

$$\text{II.} \quad \int \frac{dx}{\sqrt[3]{\log \frac{1}{x}}} = \sqrt[3]{9 \int \frac{dx}{\sqrt[3]{1-x^3}} \cdot \int \frac{x^3 dx}{\sqrt[3]{1-x^3}}};$$

the last formula because of

$$\int \frac{x^3dx}{\sqrt[3]{1-x^3}} = \frac{1}{3}\int \frac{dx}{\sqrt[3]{1-x^3}}$$

can be transformed into this one

$$\int \frac{dx}{\sqrt[3]{\log\frac{1}{x}}} = \sqrt[3]{\int \frac{dx}{\sqrt[3]{1-x^3}} \cdot \int \frac{xdx}{\sqrt[3]{1-x^3}}}$$

COROLLARY 2

§32 If, for the sake of brevity, as in my observations mentioned before[4] we put

$$\int \frac{x^{p-1}dx}{\sqrt[3]{(1-x^3)^{3-q}}} = \left(\frac{p}{q}\right)$$

and, as we did it there, for this class also set

$$\left(\frac{2}{1}\right) = \frac{\pi}{3\sin\frac{\pi}{3}} = \alpha,$$

but then put

$$\left(\frac{1}{1}\right) = \int \frac{dx}{\sqrt[3]{(1-x^3)^2}} = A,$$

it will be

$$\text{I.} \quad \int \frac{dx}{\sqrt[3]{\left(\log\frac{1}{x}\right)^2}} = \sqrt[3]{9\left(\frac{1}{1}\right)\left(\frac{2}{1}\right)} = \sqrt[3]{9\alpha A},$$

$$\text{II.} \quad \int \frac{dx}{\sqrt[3]{\left(\log\frac{1}{x}\right)^1}} = \sqrt[3]{3\left(\frac{1}{2}\right)\left(\frac{2}{2}\right)} = \sqrt[3]{\frac{3\alpha\alpha}{A}}.$$

[4]Euler refers to his paper E321 again.

COROLLARY 3

§33 Therefore, we will have for the first case

$$\int dx \sqrt[3]{\left(\log \frac{1}{x}\right)^{-2}} = \sqrt[3]{9\alpha A}, \quad \int dx \sqrt[3]{\log \frac{1}{x}} = \frac{1}{3}\sqrt[3]{9\alpha A}$$

and

$$\int dx \sqrt[3]{\left(\log \frac{1}{x}\right)^{3n+1}} = \frac{1}{3}\cdot\frac{4}{3}\cdot\frac{7}{3}\cdots\frac{3n+1}{3}\sqrt[3]{9\alpha A},$$

but for the other case on the other hand we will find

$$\int dx \sqrt[3]{\left(\log \frac{1}{x}\right)^{-1}} = \sqrt[3]{\frac{3\alpha\alpha}{A}}, \quad \int dx \sqrt[3]{\left(\log \frac{1}{x}\right)^{2}} = \frac{2}{3}\sqrt[3]{\frac{3\alpha\alpha}{A}}$$

and

$$\int dx \sqrt[3]{\left(\log \frac{1}{x}\right)^{3n-1}} = \frac{2}{3}\cdot\frac{5}{3}\cdot\frac{8}{3}\cdots\frac{3n-1}{3}\sqrt[3]{\frac{3\alpha\alpha}{A}}.$$

PROBLEM 4

While i denotes a positive integer, to define the value of the integral $\int dx \left(\log \frac{1}{x}\right)^{\frac{i}{4}-1}$, having extended the integration from $x = 0$ to $x = 1$.

SOLUTION

In the solution of the preceding problem we were led to this equation

$$\frac{\left(\int dx \left(\log \frac{1}{x}\right)^{n-1}\right)^3}{\int dx \left(\log \frac{1}{x}\right)^{3n-1}} = kk\int \frac{x^{nk-1}dx}{(1-x^k)^{1-n}} \cdot \int \frac{x^{2nk-1}dx}{(1-x^k)^{1-n}};$$

but the general formula, by setting $m = 3n$ in it, yields

$$\frac{\int dx \left(\log \frac{1}{x}\right)^{n-1} \cdot \int dx \left(\log \frac{1}{x}\right)^{3n-1}}{\int dx \left(\log \frac{1}{x}\right)^{4n-1}} = k\int \frac{x^{3nk-1}dx}{(1-x^k)^{1-n}};$$

by combining these formulas we obtain

$$\frac{\left(\int dx\left(\log\frac{1}{x}\right)^{n-1}\right)^4}{\int dx\left(\log\frac{1}{x}\right)^{4n-1}} = k^3\int\frac{x^{nk-1}dx}{(1-x^k)^{1-n}}\cdot\int\frac{x^{2nk-1}dx}{(1-x^k)^{1-n}}\cdot\int\frac{x^{3nk-1}dx}{(1-x^k)^{1-n}}.$$

Let $n=\frac{i}{4}$ and take $k=4$ and it will be

$$\frac{\int dx\left(\log\frac{1}{x}\right)^{\frac{i}{4}-1}}{\sqrt[4]{1\cdot 2\cdot 3\cdots(i-1)}} = \sqrt[4]{4^3\int\frac{x^{i-1}dx}{\sqrt[4]{(1-x^4)^{4-i}}}\cdot\int\frac{x^{2i-1}dx}{\sqrt[4]{(1-x^4)^{4-i}}}\cdot\int\frac{x^{3i-1}dx}{\sqrt[4]{(1-x^4)^{4-i}}}}.$$

COROLLARY 1

§35 So if it is $i=1$, we will have this equation

$$\int dx\sqrt[4]{\left(\log\frac{1}{x}\right)^{-3}} = \sqrt[4]{4^3\int\frac{dx}{\sqrt[4]{(1-x^4)^3}}\cdot\int\frac{xdx}{\sqrt[4]{(1-x^4)^3}}\cdot\int\frac{x^2dx}{\sqrt[4]{(1-x^4)^3}}};$$

if this expression is denoted by the letter P, it will be in general

$$\int dx\sqrt[4]{\left(\log\frac{1}{x}\right)^{4n-3}} = \frac{1}{4}\cdot\frac{5}{4}\cdot\frac{9}{4}\cdots\frac{4n-3}{4}P.$$

COROLLARY 2

§36 For the other principal case let us take $i=3$ and it will be

$$\int dx\sqrt[4]{\left(\log\frac{1}{x}\right)^{-1}} = \sqrt[4]{2\cdot 4^3\int\frac{x^2dx}{\sqrt[4]{1-x^4}}\cdot\int\frac{x^5dx}{\sqrt[4]{1-x^4}}\cdot\int\frac{x^8dx}{\sqrt[4]{1-x^4}}}$$

or after a simplification

$$\int dx\sqrt[4]{\left(\log\frac{1}{x}\right)^{-1}} = \sqrt[4]{8\int\frac{xxdx}{\sqrt[4]{1-x^4}}\cdot\int\frac{xdx}{\sqrt[4]{1-x^4}}\cdot\int\frac{dx}{\sqrt[4]{1-x^4}}};$$

if this expression is denoted by the letter Q, it will be in general

$$\int dx\sqrt[4]{\left(\log\frac{1}{x}\right)^{4n-1}} = \frac{3}{4}\cdot\frac{7}{4}\cdot\frac{11}{4}\cdots\frac{4n-1}{4}Q.$$

SCHOLIUM

§37 If we indicate the integral formula $\int \frac{x^{p-1}dx}{\sqrt[4]{(1-x^4)^{4-q}}}$ by the sign $\left(\frac{p}{q}\right)$, the solution in general will be as follows

$$\int dx \sqrt[4]{\log\left(\frac{1}{x}\right)^{i-4}} = \sqrt[4]{1\cdot 2\cdot 3\cdots(i-1)4^3\left(\frac{i}{i}\right)\left(\frac{2i}{i}\right)\left(\frac{3i}{i}\right)}$$

and for the two cases expanded before

$$P = \sqrt[4]{4^3\left(\frac{1}{1}\right)\left(\frac{2}{1}\right)\left(\frac{3}{1}\right)} \quad \text{and} \quad Q = \sqrt[4]{8\left(\frac{3}{3}\right)\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)}.$$

Now for the formulas depending on the circle let us put

$$\left(\frac{3}{1}\right) = \frac{\pi}{4\sin\frac{\pi}{4}} = \alpha \quad \text{and} \quad \left(\frac{2}{2}\right) = \frac{\pi}{4\sin\frac{2\pi}{4}} = \beta,$$

but for transcendental ones of higher order

$$\left(\frac{2}{1}\right) = \int \frac{xdx}{\sqrt[4]{(1-x^4)^3}} = \int \frac{dx}{\sqrt[2]{1-x^4}} = A,$$

on which all remaining depend; hence we will find

$$P = \sqrt[4]{4^3\frac{\alpha\alpha}{\beta}AA} \quad \text{and} \quad Q = \sqrt[4]{4\alpha\alpha\beta\frac{1}{AA}},$$

whence it is clear that it is

$$PQ = 4\alpha = \frac{\pi}{\sin\frac{\pi}{4}}.$$

But because it is $\alpha = \frac{\pi}{2\sqrt{2}}$ and $\beta = \frac{\pi}{4}$, it will be

$$P = \sqrt[4]{32\pi AA} \quad \text{and} \quad Q = \sqrt[4]{\frac{\pi^3}{8AA}} \quad \text{and} \quad \frac{P}{Q} = \frac{4A}{\sqrt{\pi}}.$$

PROBLEM 5

§38 *While i denotes a positive integer, to define the value of the integral* $\int dx \sqrt[5]{\left(\log\frac{1}{x}\right)^{i-5}}$, *having extended the integration from* $x = 0$ *to* $x = 1$.

SOLUTION

From the preceding solutions it is already perspicuous that for this case one will obtain this formula at the end

$$\frac{\int dx \sqrt[5]{\left(\log \frac{1}{x}\right)^{i-5}}}{\sqrt[5]{1 \cdot 2 \cdot 3 \cdots (i-1)}} = \sqrt[5]{5^4 \int \frac{x^{i-1}dx}{\sqrt[5]{(1-x^5)^{5-i}}} \cdot \int \frac{x^{2i-1}dx}{\sqrt[5]{(1-x^5)^{5-i}}} \cdot \int \frac{x^{3i-1}dx}{\sqrt[5]{(1-x^5)^{5-i}}} \cdot \int \frac{x^{4i-1}dx}{\sqrt[5]{(1-x^5)^{5-i}}}},$$

which integral formulas are to be referred to the fifth class introduced in my dissertation mentioned above[5]. Hence if in the same way as it was done there the sign $\left(\frac{p}{q}\right)$ denotes this formula $\int \frac{x^{p-1}dx}{\sqrt[5]{(1-x^5)^{5-q}}}$, the value in question can be more conveniently expressed in such a way that it is

$$\int dx \sqrt[5]{\left(\log \frac{1}{x}\right)^{i-5}} = \sqrt[5]{1 \cdot 2 \cdot 3 \cdots (i-1) 5^4 \left(\frac{i}{i}\right) \left(\frac{2i}{i}\right) \left(\frac{3i}{i}\right) \left(\frac{4i}{i}\right)};$$

here it indeed suffices to have assigned values smaller than five to i; for, if the numerators exceed five it is to be noted that it is

$$\left(\frac{5+m}{i}\right) = \frac{m}{m+i} \left(\frac{m}{i}\right),$$

but then further

$$\left(\frac{10+m}{i}\right) = \frac{m}{m+i} \cdot \frac{m+5}{m+i+5} \left(\frac{m}{i}\right),$$

$$\left(\frac{15+m}{i}\right) = \frac{m}{m+i} \cdot \frac{m+5}{m+i+5} \cdot \frac{m+10}{m+i+10} \left(\frac{m}{i}\right).$$

Furthermore, for this class two formulas indeed involve the quadrature of the circle; these formulas are

$$\left(\frac{4}{1}\right) = \frac{\pi}{5 \sin \frac{\pi}{5}} = \alpha \quad \text{and} \quad \left(\frac{3}{2}\right) = \frac{\pi}{5 \sin \frac{2\pi}{5}} = \beta,$$

but then two contain higher quadratures, which we want to put

[5]Euler again refers to his paper "Observationes circa integralia formularum $\int x^{p-1}dx(1-x^n)^{\frac{q}{n}-1}$ posito post integrationem $x=1$". This is paper E321 in the Eneström-Index.

$$\left(\frac{3}{1}\right) = \int \frac{xxdx}{\sqrt[5]{(1-x^5)^4}} = \int \frac{dx}{\sqrt[5]{(1-x^5)^2}} = A \quad \text{and} \quad \left(\frac{2}{2}\right) = \int \frac{xdx}{\sqrt[5]{(1-x^5)^3}} = B,$$

and using these I assigned the values of all remaining formulas of this class[6], namely

$$\begin{array}{lllll}
\left(\frac{5}{1}\right) = 1, & \left(\frac{5}{2}\right) = \frac{1}{2}, & \left(\frac{5}{3}\right) = \frac{1}{3} & \left(\frac{5}{4}\right) = \frac{1}{4}, & \left(\frac{5}{5}\right) = \frac{1}{5}; \\
\left(\frac{4}{1}\right) = \alpha, & \left(\frac{4}{2}\right) = \frac{\beta}{A}, & \left(\frac{4}{3}\right) = \frac{\beta}{2B} & \left(\frac{4}{4}\right) = \frac{\alpha}{3A}; & \\
\left(\frac{3}{1}\right) = A, & \left(\frac{3}{2}\right) = \beta, & \left(\frac{3}{3}\right) = \frac{\beta\beta}{\alpha B}; & & \\
\left(\frac{2}{1}\right) = \frac{\alpha B}{\beta}, & \left(\frac{2}{2}\right) = B; & & & \\
\left(\frac{1}{1}\right) = \frac{\alpha A}{\beta}. & & & &
\end{array}$$

COROLLARY 1

Having taken the exponent $i = 1$, it will be

$$\int dx \sqrt[5]{\left(\log\frac{1}{x}\right)^{-4}} = \sqrt[5]{5^4 \left(\frac{1}{1}\right)\left(\frac{2}{1}\right)\left(\frac{3}{1}\right)\left(\frac{4}{1}\right)} = \sqrt[5]{5^4 \frac{\alpha^3}{\beta^2} A^2 B},$$

whence we conclude in general that, while n denotes a positive integer, it is

$$\int dx \sqrt[5]{\left(\log\frac{1}{x}\right)^{5n-4}} = \frac{1}{5}\cdot\frac{6}{5}\cdot\frac{11}{5}\cdots\frac{5n-4}{5}\sqrt[5]{5^4 \frac{\alpha^3}{\beta} A^2 B}.$$

COROLLARY 2

§40 Now let it be $i = 2$, and because then this equation results

$$\int dx \sqrt[5]{\left(\log\frac{1}{x}\right)^{-3}} = \sqrt[5]{5^4 \left(\frac{2}{2}\right)\left(\frac{4}{2}\right)\left(\frac{6}{2}\right)\left(\frac{8}{2}\right)},$$

[6]Euler took the following list out of E321.

because of

$$\binom{6}{2} = \frac{1}{3}\binom{1}{2} = \frac{1}{3}\binom{2}{1} \quad \text{and} \quad \binom{8}{2} = \frac{3}{3}\binom{3}{2}$$

the left hand side will be

$$\sqrt[5]{5^3\binom{2}{2}\binom{4}{2}\binom{2}{1}\binom{3}{2}} = \sqrt[5]{5^3\alpha\beta\frac{BB}{A}}$$

and in general

$$\int dx\sqrt[5]{\left(\log\frac{1}{x}\right)^{5n-3}} = \frac{2}{5}\cdot\frac{7}{5}\cdot\frac{12}{5}\cdots\frac{5n-3}{5}\sqrt[5]{5^3\alpha\beta\frac{BB}{A}}.$$

COROLLARY 3

§41 Let $i = 3$ and the form found

$$\int dx\sqrt[5]{\left(\log\frac{1}{x}\right)^{-2}} = \sqrt[5]{2\cdot 5^4\binom{3}{3}\binom{6}{3}\binom{9}{3}\binom{12}{3}}$$

because of

$$\binom{6}{3} = \frac{1}{4}\binom{3}{1}, \quad \binom{9}{3} = \frac{4}{7}\binom{4}{3}, \quad \binom{12}{3} = \frac{2}{5}\cdot\frac{7}{10}\binom{3}{2}$$

changes to

$$\sqrt[5]{2\cdot 5^2\binom{3}{3}\binom{3}{1}\binom{4}{3}\binom{3}{2}} = \sqrt[5]{5^2\frac{\beta^4}{\alpha}\cdot\frac{A}{BB}},$$

whence it is concluded that it is in general

$$\int dx\sqrt[5]{\left(\log\frac{1}{x}\right)^{5n-2}} = \frac{3}{5}\cdot\frac{8}{5}\cdot\frac{13}{5}\cdots\frac{5n-2}{5}\sqrt[5]{5^2\frac{\beta^4}{\alpha}\cdot\frac{A}{BB}}.$$

COROLLARY 4

§42 Finally, for $i = 4$ our equation

$$\int dx \sqrt[5]{\left(\log \frac{1}{x}\right)^{-1}} = \sqrt[5]{6 \cdot 5^4 \left(\frac{4}{4}\right)\left(\frac{8}{4}\right)\left(\frac{12}{4}\right)\left(\frac{16}{4}\right)}$$

because of

$$\left(\frac{8}{4}\right) = \frac{3}{7}\left(\frac{4}{3}\right), \quad \left(\frac{12}{4}\right) = \frac{2}{6} \cdot \frac{7}{11}\left(\frac{4}{2}\right), \quad \left(\frac{16}{4}\right) = \frac{1}{5} \cdot \frac{6}{10} \cdot \frac{11}{15}\left(\frac{4}{1}\right)$$

will be transformed into this form

$$\sqrt[5]{6 \cdot 5 \left(\frac{4}{4}\right)\left(\frac{4}{3}\right)\left(\frac{4}{2}\right)\left(\frac{4}{1}\right)} = \sqrt[5]{5\frac{\alpha\alpha\beta\beta}{AAB}},$$

so that it is in general

$$\int dx \sqrt[5]{\left(\log \frac{1}{x}\right)^{5n-1}} = \frac{4}{5} \cdot \frac{9}{5} \cdot \frac{14}{5} \cdots \frac{5n-1}{5} \sqrt[5]{5\alpha\alpha\beta\beta\frac{1}{AAB}}.$$

SCHOLIUM

§43 If we represent the value of the integral formula $\int dx \left(\log \frac{1}{x}\right)^{\lambda}$ by the sign $[\lambda]$, the cases expanded up to now yield

$$\left[-\frac{4}{5}\right] = \sqrt[5]{5^4\frac{\alpha^3}{\beta^2} \cdot A^2B}, \qquad \left[+\frac{1}{5}\right] = \frac{1}{5}\sqrt[5]{5^4\frac{\alpha^3}{\beta^2} \cdot A^2B},$$

$$\left[-\frac{3}{5}\right] = \sqrt[5]{5^3\alpha\beta \cdot \frac{BB}{A}}, \qquad \left[+\frac{2}{5}\right] = \frac{2}{5}\sqrt[5]{5^3\alpha\beta\frac{BB}{A}},$$

$$\left[-\frac{2}{5}\right] = \sqrt[5]{5^2\frac{\beta^4}{\beta} \cdot \frac{A}{BB}}, \qquad \left[+\frac{3}{5}\right] = \frac{3}{5}\sqrt[5]{5^2\frac{\beta^4}{\alpha} \cdot \frac{A}{BB}},$$

$$\left[-\frac{1}{5}\right] = \sqrt[5]{5\alpha^2\beta^2 \cdot \frac{1}{AAB}}, \qquad \left[+\frac{4}{5}\right] = \frac{4}{5}\sqrt[5]{5\alpha^2\beta^2 \cdot \frac{1}{AAB}},$$

whence by combining two, whose indices add up to 0, we conclude

$$\begin{aligned}
\left[+\frac{1}{5}\right]\cdot\left[-\frac{1}{5}\right] &= \alpha \;\; = \frac{\pi}{5\sin\frac{\pi}{5}},\\
\left[+\frac{2}{5}\right]\cdot\left[-\frac{2}{5}\right] &= 2\beta = \frac{2\pi}{5\sin\frac{2\pi}{5}},\\
\left[+\frac{3}{5}\right]\cdot\left[-\frac{3}{5}\right] &= 3\beta = \frac{3\pi}{5\sin\frac{3\pi}{5}},\\
\left[+\frac{4}{5}\right]\cdot\left[-\frac{4}{5}\right] &= 4\alpha = \frac{\pi}{5\sin\frac{4\pi}{5}}.
\end{aligned}$$

But from the preceding problem in like manner we deduce:

$$\begin{aligned}
\left[-\frac{3}{4}\right] &= P \;= \sqrt[4]{4^3\frac{\alpha\alpha}{\beta}\cdot AA}, \quad & \left[+\frac{1}{4}\right] &= \frac{1}{4}\sqrt[4]{4^3\frac{\alpha\alpha}{\beta}\cdot AA},\\
\left[-\frac{1}{4}\right] &= Q = \sqrt[4]{4\alpha\alpha\beta\cdot\frac{1}{AA}}, \quad & \left[+\frac{3}{4}\right] &= \frac{3}{4}\sqrt[4]{4\alpha\alpha\beta\cdot\frac{1}{AA}}
\end{aligned}$$

and hence

$$\begin{aligned}
\left[+\frac{1}{4}\right]\cdot\left[-\frac{1}{4}\right] &= \alpha \;\; = \frac{\pi}{4\sin\frac{\pi}{4}},\\
\left[+\frac{3}{4}\right]\cdot\left[-\frac{3}{4}\right] &= 3\alpha = \frac{3\pi}{4\sin\frac{3\pi}{4}},
\end{aligned}$$

whence in general we obtain this theorem that it is

$$[\lambda]\cdot[-\lambda] = \frac{\lambda\pi}{\sin\lambda\pi};$$

the reason this can be given from the interpolation method explained some time ago[7] as follows. Because it is

$$[\lambda] = \frac{1^{1-\lambda}\cdot 2^{\lambda}}{1+\lambda}\cdot\frac{2^{1-\lambda}\cdot 3^{\lambda}}{2+\lambda}\cdot\frac{3^{1-\lambda}\cdot 4^{\lambda}}{3+\lambda}\cdot \text{etc.},$$

[7]Euler explains the interpolation method he talks about here also in E19.

it will be

$$[-\lambda] = \frac{1^{1+\lambda} \cdot 2^{-\lambda}}{1-\lambda} \cdot \frac{2^{1+\lambda} \cdot 3^{-\lambda}}{2-\lambda} \cdot \frac{3^{1+\lambda} \cdot 4^{-\lambda}}{3-\lambda} \cdot \text{etc.}$$

and hence

$$[\lambda] \cdot [-\lambda] = \frac{1 \cdot 1}{1-\lambda\lambda} \cdot \frac{2 \cdot 2}{4-\lambda\lambda} \cdot \frac{3 \cdot 3}{9-\lambda\lambda} \cdot \text{etc.} = \frac{\lambda\pi}{\sin \lambda\pi},$$

as I demonstrated elsewhere[8].

Problem 6 - General Problem

§44 *If the letters i and n denote positive integers, to define the value of the integral*

$$\int dx \left(\log \frac{1}{x}\right)^{\frac{i-n}{n}} \quad or \quad \int dx \sqrt[n]{\left(\log \frac{1}{x}\right)^{i-n}}$$

having extended the integration from $x = 0$ to $x = 1$.

Solution

The method up to now will exhibit the value in question expressed by quadratures of algebraic curves in the following way

$$\frac{\int dx \sqrt[n]{\left(\log \frac{1}{x}\right)^{i-n}}}{\sqrt[n]{1 \cdot 2 \cdot 3 \cdots (i-1)}} = \sqrt[n]{n^{n-1} \int \frac{x^{i-1}dx}{\sqrt[n]{(1-x^n)^{n-i}}} \cdot \int \frac{x^{2i-1}dx}{\sqrt[n]{(1-x^n)^{n-i}}} \cdots \int \frac{x^{(n-1)i-1}dx}{\sqrt[n]{(1-x^n)^{n-i}}}}$$

Hence, if, for the sake of brevity, we denote the integral formula $\int \frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}$ by this character $\left(\frac{p}{q}\right)$, but on the other hand the formula $\int dx \sqrt[n]{\left(\log \frac{1}{x}\right)^m}$ by this character $\left[\frac{m}{n}\right]$ so that $\left[\frac{m}{n}\right]$ denotes the value of this indefinite product $1 \cdot 2 \cdot 3 \cdots z$, while $z = \frac{m}{n}$, the in value question for will expressed more succinctly as follows

[8]Euler proved this relation in his paper "Methodus facilis computandi angulorum sinus ac tangentes tam naturales quam artificiales". This is paper E128 in the Eneström-Index.

$$\left[\frac{i-n}{n}\right] = \sqrt[n]{1\cdot 2\cdot 3\cdots (i-1)n^{n-1}\left(\frac{i}{i}\right)\left(\frac{2i}{i}\right)\left(\frac{3i}{i}\right)\cdots\left(\frac{ni-i}{i}\right)},$$

whence it is also concluded

$$\left[\frac{i}{n}\right] = \frac{i}{n}\sqrt[n]{1\cdot 2\cdot 3\cdots (i-1)n^{n-1}\left(\frac{i}{i}\right)\left(\frac{2i}{i}\right)\left(\frac{3i}{i}\right)\cdots\left(\frac{ni-i}{i}\right)}.$$

Here it will always suffice to have taken the number i smaller than n, because it is known for larger numbers that it is

$$\left[\frac{i+n}{n}\right] = \frac{i+n}{n}\left[\frac{i}{n}\right], \quad \text{in the same way} \quad \left[\frac{i+2n}{n}\right] = \frac{i+n}{n}\cdot\frac{i+2n}{n}\left[\frac{i}{n}\right] \quad \text{etc.},$$

and so the whole investigation is hence reduced to those cases, in which the numerator i of the fraction $\frac{i}{n}$ is smaller than the denominator n. In addition, it will be helpful to have noted the following properties of the integral formulas

$$\int \frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}} = \left(\frac{p}{q}\right):$$

I. The letters p and q are interchangeable so that it is

$$\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right).$$

II. If one of the two numbers p or q is equal to the exponent n, the value of the integral formula will be algebraic, namely

$$\left(\frac{n}{p}\right) = \left(\frac{p}{n}\right) = \frac{1}{p} \quad \text{or} \quad \left(\frac{n}{q}\right) = \left(\frac{q}{n}\right) = \frac{1}{q}.$$

III. If the sum of the numbers $p+q$ is equal to the exponent n, the value of the integral formula $\left(\frac{p}{q}\right)$ can be exhibited by means of the quadrature of the circle, because it is

$$\left(\frac{p}{n-p}\right) = \left(\frac{n-p}{p}\right) = \frac{\pi}{n\sin\frac{p\pi}{n}} \quad \text{and} \quad \left(\frac{q}{n-q}\right) = \left(\frac{n-q}{q}\right) = \frac{\pi}{n\sin\frac{q\pi}{n}}.$$

IV. If one of the numbers p or q is greater than the exponent n, the integral formula $\left(\frac{p}{q}\right)$ can be reduced to another one, whose terms are smaller than n; this is achieved using this reduction

$$\left(\frac{p+n}{q}\right) = \frac{p}{p+q}\left(\frac{p}{q}\right).$$

V. There is a relation among many of these integral formulas of such a kind that it is

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right) = \left(\frac{p}{r}\right)\left(\frac{p+r}{q}\right) = \left(\frac{q}{r}\right)\left(\frac{q+r}{p}\right);$$

by means of this relation all reductions are found I gave in my observations on these formulas[9].

COROLLARY 1

§45 If we accommodate the found formula to the single cases in this way by means of reduction IV, we will be able to exhibit them in the most simple way as follows. And for the case $n = 2$, in which no further reduction is necessary, we will have

$$\left[\frac{1}{2}\right] = \frac{1}{2}\sqrt[2]{2\left(\frac{1}{1}\right)} = \frac{1}{2}\sqrt[2]{\frac{\pi}{\sin\frac{\pi}{2}}} = \frac{1}{2}\sqrt{\pi}.$$

COROLLARY 2

§46 For the cases $n = 3$ we will have these reductions

$$\left[\frac{1}{3}\right] = \frac{1}{3}\sqrt[3]{3^2\left(\frac{1}{1}\right)\left(\frac{2}{1}\right)}$$

$$\left[\frac{2}{3}\right] = \frac{2}{3}\sqrt[3]{3\cdot 1\left(\frac{2}{2}\right)\left(\frac{1}{2}\right)}.$$

[9]Euler is again referring to E321.

COROLLARY 3

§47 For the case $n = 4$ one obtains these three reductions

$$\left[\frac{1}{4}\right] = \frac{1}{4}\sqrt[4]{4^3\left(\frac{1}{1}\right)\left(\frac{2}{1}\right)\left(\frac{3}{1}\right)},$$

$$\left[\frac{2}{4}\right] = \frac{2}{4}\sqrt[4]{4^2\cdot 2\left(\frac{2}{2}\right)^2\left(\frac{4}{2}\right)} = \frac{1}{2}\sqrt[2]{4\left(\frac{2}{2}\right)}$$

because of $\left(\frac{4}{2}\right) = \frac{1}{2}$,

$$\left[\frac{3}{4}\right] = \frac{3}{4}\sqrt{4\cdot 1\cdot 2\left(\frac{3}{3}\right)\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)};$$

because in the second equation it is $\left(\frac{2}{2}\right) = \left(\frac{4-2}{2}\right) = \frac{\pi}{4}$, it will be as before, of course

$$\left[\frac{2}{4}\right] = \left[\frac{1}{2}\right] = \frac{1}{2}\sqrt{\pi}.$$

COROLLARY 4

§48 Now let n be $= 5$ and these four reductions result

$$\left[\frac{1}{5}\right] = \frac{1}{5}\sqrt[5]{5^4\left(\frac{1}{1}\right)\left(\frac{2}{1}\right)\left(\frac{3}{1}\right)\left(\frac{4}{1}\right)},$$

$$\left[\frac{2}{5}\right] = \frac{2}{5}\sqrt[5]{5^3\cdot 2\left(\frac{2}{2}\right)\left(\frac{4}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)},$$

$$\left[\frac{3}{5}\right] = \frac{3}{5}\sqrt[5]{5^2\cdot 1\cdot 2\left(\frac{3}{3}\right)\left(\frac{1}{3}\right)\left(\frac{4}{3}\right)\left(\frac{2}{3}\right)},$$

$$\left[\frac{4}{5}\right] = \frac{4}{5}\sqrt[5]{5\cdot 1\cdot 2\cdot 3\left(\frac{4}{4}\right)\left(\frac{3}{4}\right)\left(\frac{2}{4}\right)\left(\frac{1}{4}\right)}.$$

COROLLARY 5

§49 Let $n = 6$ and we will have these reductions

$$\left[\frac{1}{6}\right] = \frac{1}{6}\sqrt[6]{6^5\left(\frac{1}{1}\right)\left(\frac{2}{1}\right)\left(\frac{3}{1}\right)\left(\frac{4}{1}\right)\left(\frac{5}{1}\right)},$$

$$\left[\frac{2}{4}\right] = \frac{2}{6}\sqrt[6]{6^4\cdot 2\left(\frac{2}{2}\right)^2\left(\frac{4}{2}\right)^2\left(\frac{6}{2}\right)} = \frac{1}{3}\sqrt[3]{6^2\left(\frac{3}{2}\right)\left(\frac{4}{2}\right)},$$

$$\left[\frac{3}{6}\right] = \frac{3}{6}\sqrt[6]{6^3\cdot 3\cdot 3\left(\frac{3}{3}\right)^3\left(\frac{6}{3}\right)^2} = \frac{1}{2}\sqrt[2]{6\left(\frac{3}{3}\right)},$$

$$\left[\frac{4}{6}\right] = \frac{4}{6}\sqrt[8]{6^2\cdot 2\cdot 4\cdot 2\left(\frac{4}{4}\right)^2\left(\frac{2}{4}\right)^2\left(\frac{6}{4}\right)} = \frac{2}{3}\sqrt[3]{6\cdot 2\left(\frac{4}{4}\right)\left(\frac{2}{4}\right)},$$

$$\left[\frac{5}{6}\right] = \frac{5}{6}\sqrt[6]{6\cdot 1\cdot 2\cdot 3\cdot 4\left(\frac{5}{5}\right)\left(\frac{4}{5}\right)\left(\frac{3}{5}\right)\left(\frac{2}{5}\right)\left(\frac{1}{5}\right)}.$$

COROLLARY 6

§50 For $n = 7$ the following six equations arise

$$\left[\frac{1}{7}\right] = \frac{1}{7}\sqrt[7]{7^6\left(\frac{1}{1}\right)\left(\frac{2}{1}\right)\left(\frac{3}{1}\right)\left(\frac{4}{1}\right)\left(\frac{5}{1}\right)\left(\frac{6}{1}\right)},$$

$$\left[\frac{2}{7}\right] = \frac{2}{7}\sqrt[7]{7^5\cdot 1\left(\frac{2}{2}\right)\left(\frac{4}{2}\right)\left(\frac{6}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)},$$

$$\left[\frac{3}{7}\right] = \frac{3}{7}\sqrt[7]{7^4\cdot 1\cdot 2\left(\frac{3}{3}\right)\left(\frac{6}{3}\right)\left(\frac{2}{3}\right)\left(\frac{5}{3}\right)\left(\frac{1}{3}\right)\left(\frac{4}{3}\right)},$$

$$\left[\frac{4}{7}\right] = \frac{4}{7}\sqrt[7]{7^3\cdot 1\cdot 2\cdot 3\left(\frac{4}{4}\right)\left(\frac{1}{4}\right)\left(\frac{5}{4}\right)\left(\frac{2}{4}\right)\left(\frac{6}{4}\right)\left(\frac{3}{4}\right)},$$

$$\left[\frac{5}{7}\right] = \frac{5}{7}\sqrt[7]{7^2\cdot 1\cdot 2\cdot 3\cdot 4\left(\frac{5}{5}\right)\left(\frac{3}{5}\right)\left(\frac{1}{5}\right)\left(\frac{6}{5}\right)\left(\frac{4}{5}\right)\left(\frac{2}{5}\right)},$$

$$\left[\frac{6}{7}\right] = \frac{6}{7}\sqrt[7]{7\cdot 1\cdot 2\cdot 3\cdot 4\cdot 5\left(\frac{6}{6}\right)\left(\frac{5}{6}\right)\left(\frac{4}{6}\right)\left(\frac{3}{6}\right)\left(\frac{2}{6}\right)\left(\frac{1}{6}\right)}.$$

COROLLARY 7

§51 Now let n be $=8$ and one will get to these seven reductions

$$\left[\frac{1}{8}\right]=\frac{1}{8}\sqrt[8]{8^7\left(\frac{1}{1}\right)\left(\frac{2}{1}\right)\left(\frac{3}{1}\right)\left(\frac{4}{1}\right)\left(\frac{5}{1}\right)\left(\frac{6}{1}\right)\left(\frac{7}{1}\right)},$$

$$\left[\frac{2}{8}\right]=\frac{2}{8}\sqrt[8]{8^6\cdot 2\left(\frac{2}{2}\right)^2\left(\frac{4}{2}\right)^2\left(\frac{6}{2}\right)^2\left(\frac{8}{2}\right)}=\frac{1}{4}\sqrt[4]{8^3\left(\frac{2}{2}\right)\left(\frac{4}{2}\right)\left(\frac{6}{2}\right)},$$

$$\left[\frac{3}{8}\right]=\frac{3}{8}\sqrt[8]{8^5\cdot 1\cdot 2\left(\frac{3}{3}\right)\left(\frac{6}{3}\right)\left(\frac{1}{3}\right)\left(\frac{4}{3}\right)\left(\frac{7}{3}\right)\left(\frac{2}{3}\right)\left(\frac{5}{3}\right)},$$

$$\left[\frac{4}{8}\right]=\frac{4}{8}\sqrt[8]{8^4\cdot 4\cdot 4\cdot 4\left(\frac{4}{4}\right)^4\left(\frac{8}{4}\right)^3}=\frac{1}{2}\sqrt[2]{8\left(\frac{4}{4}\right)},$$

$$\left[\frac{5}{8}\right]=\frac{5}{8}\sqrt[8]{8^3\cdot 1\cdot 2\cdot 3\cdot 4\left(\frac{5}{5}\right)\left(\frac{2}{5}\right)\left(\frac{7}{5}\right)\left(\frac{4}{5}\right)\left(\frac{1}{5}\right)\left(\frac{6}{5}\right)\left(\frac{3}{5}\right)},$$

$$\left[\frac{6}{8}\right]=\frac{6}{8}\sqrt[8]{8^2\cdot 4\cdot 2\cdot 6\cdot 4\cdot 2\left(\frac{6}{6}\right)^2\left(\frac{4}{6}\right)^2\left(\frac{2}{6}\right)^2\left(\frac{8}{6}\right)}=\frac{3}{4}\sqrt[4]{8\cdot 2\cdot 4\left(\frac{6}{6}\right)\left(\frac{4}{6}\right)\left(\frac{2}{6}\right)},$$

$$\left[\frac{7}{8}\right]=\frac{7}{8}\sqrt[8]{8\cdot 1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\left(\frac{7}{7}\right)\left(\frac{6}{7}\right)\left(\frac{5}{7}\right)\left(\frac{4}{7}\right)\left(\frac{3}{7}\right)\left(\frac{2}{7}\right)\left(\frac{1}{7}\right)}.$$

SCHOLIUM

§52 It would be superfluous to expand these cases any further, because the structure of these formulas is already seen very clearly from the ones listed. If the numbers m and n are coprime in the propounded formula $\left[\frac{m}{n}\right]$, the rule is manifest, because

$$\left[\frac{m}{n}\right]=\frac{m}{n}\sqrt[n]{n^{n-m}\cdot 1\cdot 2\cdots(m-1)\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n-1}{m}\right)};$$

but if these numbers m and n have a common divisor, it will indeed be useful, to reduce this fraction $\frac{m}{n}$ to the smallest form and extract the value in question from the preceding cases; nevertheless, the operation can also be done as follows. Because the expression in question certainly has this form

$$\left[\frac{m}{n}\right] = \frac{m}{n}\sqrt[n]{n^{n-m}PQ},$$

where Q is the product of the $n-1$ integral formulas, P on the other hand the product of some absolute numbers, in order to find that product Q just continue this series of formulas $\left(\frac{m}{m}\right)\left(\frac{2m}{m}\right)\left(\frac{3m}{m}\right)$etc., until the numerator exceeds the exponent n, and instead of this numerator write its excess over n; if this excess is put $=\alpha$ that our formula is $\left(\frac{\alpha}{m}\right)$, this numerator α will give a factor of a product P; then hence further set this series of formulas $\left(\frac{\alpha}{m}\right)\left(\frac{\alpha+m}{m}\right)\left(\frac{\alpha+2m}{m}\right)$etc., until one again gets to a numerator greater than the exponent n and the formula $\left(\frac{n+\beta}{m}\right)$ emerges; instead of this formula one then has to write $\left(\frac{\beta}{m}\right)$, and hence the factor β is introduced into the product and like this one has to continue, until $n-1$ formulas for Q will have emerged.

To understand these operations more easily, let us expand the case of the formula

$$\left[\frac{9}{12}\right] = \frac{9}{12}\sqrt[12]{12^3PQ}$$

in this way, where the letters P and Q is are found as follows:

$$\begin{array}{lccccccccccc}
\text{for}\quad Q\ldots & \left(\frac{9}{9}\right) & \left(\frac{6}{9}\right) & \left(\frac{3}{9}\right) & \left(\frac{12}{9}\right) & \left(\frac{9}{9}\right) & \left(\frac{6}{9}\right) & \left(\frac{3}{9}\right) & \left(\frac{12}{9}\right) & \left(\frac{9}{9}\right) & \left(\frac{6}{9}\right) & \left(\frac{3}{9}\right), \\
\text{for}\quad P\ldots & & 6 & \cdot\ 3 & & 9 & \cdot\ 6 & \cdot\ 3 & & 9 & \cdot\ 6 & \cdot\ 3
\end{array}$$

and so one finds

$$Q = \left(\frac{9}{9}\right)^3\left(\frac{6}{9}\right)^3\left(\frac{3}{9}\right)^3\left(\frac{12}{9}\right)^2 \quad \text{and} \quad P = 6^3\cdot 3^3\cdot 9^2.$$

Because it is $\left(\frac{12}{9}\right)=\frac{1}{9}$, it is $PQ = 6^3\cdot 3^3\left(\frac{9}{9}\right)^3\left(\frac{6}{9}\right)^3\left(\frac{3}{9}\right)^3$ and hence

$$\left[\frac{9}{12}\right] = \frac{3}{4}\sqrt[4]{12\cdot 6\cdot 3\left(\frac{9}{9}\right)\left(\frac{6}{9}\right)\left(\frac{3}{9}\right)}.$$

THEOREM

§53 *No matter which positive numbers are indicated by the letters m and n, in the notation introduced and explained before it will always be*

$$\left[\frac{m}{n}\right] = \frac{m}{m}\sqrt[n]{n^{n-m}\cdot 1\cdot 2\cdot 3\cdots(m-1)\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n-1}{m}\right)}.$$

PROOF

For the cases, in which m and n are coprime numbers, the validity of this theorem was shown in the preceding theorems; but that it also holds, if those numbers m and n have a common divisor, is not evident from that theorem; but since the formula was already proved to be true in the cases, in which m and n are mutually prime, one it is natural to conclude that this theorem is true in general. I am completely aware that this kind to conclude something is completely unusual and has to seem suspect to most people. In order to clear those doubts doubts, because for the cases, in which the numbers m and n are composite, we obtained two expressions, it will be useful to have shown the agreement for the cases explained before. But the case $m = n$ already is a huge confirmation, in which case our formula manifestly becomes $= 1$.

COROLLARY 1

§54 The first case requiring a proof of the agreement is that one, in which it is $m = 2$ and $n = 4$, for which we found above (§47)

$$\left[\frac{2}{4}\right] = \frac{2}{4}\sqrt[4]{4^2\left(\frac{2}{2}\right)^2};$$

but now via the theorem it is

$$\left[\frac{2}{4}\right] = \frac{2}{4}\sqrt[4]{4^2\cdot 1\left(\frac{1}{2}\right)\left(\frac{2}{2}\right)\left(\frac{3}{2}\right)},$$

where by comparison it is

$$\left(\frac{2}{2}\right) = \left(\frac{1}{2}\right)\left(\frac{3}{2}\right),$$

whose validity was confirmed in my observations mentioned[10] above.

[10]Euler again refers to his paper "Observationes circa integralia formularum $\int x^{p-1}dx(1-x^n)^{\frac{q}{n}-1}$ posito post integrationem $x = 1$". This is paper E321 in the Eneström-Index.

Corollary 2

§55 If it is $m = 2$ and $n = 6$, using the results derived above (§49) it is

$$\left[\frac{2}{6}\right] = \frac{2}{6}\sqrt[6]{6^4\left(\frac{2}{2}\right)^2\left(\frac{4}{2}\right)^2};$$

now on the other hand by means of the theorem it is

$$\left[\frac{2}{6}\right] = \frac{2}{6}\sqrt[6]{6^4\cdot 1\left(\frac{1}{2}\right)\left(\frac{2}{2}\right)\left(\frac{3}{2}\right)\left(\frac{4}{2}\right)\left(\frac{5}{2}\right)}$$

and therefore it has to be

$$\left(\frac{2}{2}\right)\left(\frac{4}{2}\right) = \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right),$$

whose validity is clear for the same reasons.

Corollary 3

§56 If it is $m = 3$ and $n = 6$, one gets to this equation

$$\left(\frac{3}{3}\right)^2 = 1\cdot 2\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)\left(\frac{4}{3}\right)\left(\frac{5}{3}\right);$$

but if $m = 4$ and $n = 6$, in like manner it is

$$2^2\left(\frac{4}{4}\right)\left(\frac{2}{4}\right) = 1\cdot 2\cdot 3\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)\left(\frac{5}{4}\right)$$

or

$$\left(\frac{4}{4}\right)\left(\frac{2}{4}\right) = \frac{3}{2}\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)\left(\frac{5}{4}\right),$$

which is also detected to be true.

Corollary 4

§57 The case $m = 2$ and $n = 8$ yields this equality

$$\left(\frac{2}{2}\right)\left(\frac{4}{2}\right)\left(\frac{6}{2}\right) = \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\left(\frac{7}{2}\right),$$

but the case $m = 4$ and $n = 8$ this one

$$\left(\frac{4}{4}\right)^2 = 1 \cdot 2 \cdot 3 \left(\frac{1}{4}\right)\left(\frac{2}{4}\right)\left(\frac{3}{4}\right)\left(\frac{5}{4}\right)\left(\frac{6}{4}\right)\left(\frac{7}{4}\right)$$

and finally the case $m = 6$ and $n = 8$ gives this equation

$$2 \cdot 4 \left(\frac{6}{6}\right)\left(\frac{4}{6}\right)\left(\frac{2}{6}\right) = 1 \cdot 3 \cdot 5 \left(\frac{1}{6}\right)\left(\frac{3}{6}\right)\left(\frac{5}{6}\right)\left(\frac{7}{6}\right),$$

which is also true.

SCHOLIUM

§58 But if in general the numbers m and n have the common factor 2 and the propounded formula is $\left[\frac{2m}{2n}\right] = \left[\frac{m}{n}\right]$, because it is

$$\left[\frac{m}{n}\right] = \frac{m}{n}\sqrt[n]{n^{n-m} \cdot 1 \cdot 2 \cdot 3 \cdots (m-1)\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n-1}{m}\right)},$$

after having reduced the same to the exponent $2n$, it will be

$$\frac{m}{n}\sqrt[2n]{2n^{2n-2m} \cdot 2^2 \cdot 4^2 \cdot 6^2 \cdots (2m-2)^2 \left(\frac{2}{2m}\right)^2\left(\frac{4}{2m}\right)^2\left(\frac{6}{2m}\right)^2\cdots\left(\frac{2n-2}{2m}\right)^2}.$$

By the theorem the same expression on the other hand becomes

$$\frac{m}{n}\sqrt[2n]{2n^{2n-2m} \cdot 1 \cdot 2 \cdot 3 \cdots (2m-1)\left(\frac{1}{2m}\right)\left(\frac{2}{2m}\right)\left(\frac{3}{2m}\right)\cdots\left(\frac{2n-1}{2m}\right)},$$

whence for the exponent $2n$ it will be

$$\begin{aligned}&2 \cdot 4 \cdot 6 \cdots (2m-2)\left(\frac{2}{2m}\right)\left(\frac{4}{2m}\right)\left(\frac{6}{2m}\right)\cdots\left(\frac{2n-2}{2m}\right)\\ &= 1 \cdot 3 \cdot 5 \cdots (2m-1)\left(\frac{1}{2m}\right)\left(\frac{3}{2m}\right)\left(\frac{5}{2m}\right)\cdots\left(\frac{2n-1}{2m}\right).\end{aligned}$$

If in the same way the common divisor is 3, one will find for the exponent $3n$

$$3^2 \cdot 6^2 \cdot 9^2 \cdots (3m-3)^2 \left(\frac{3}{3m}\right)^2 \left(\frac{6}{3m}\right)^2 \left(\frac{9}{3m}\right)^2 \cdots \left(\frac{3n-3}{3m}\right)^2$$
$$= 1 \cdot 2 \cdot 4 \cdot 5 \cdots (3m-2)(3m-1) \left(\frac{1}{3m}\right) \left(\frac{2}{3m}\right) \left(\frac{4}{3m}\right) \left(\frac{5}{3m}\right) \cdots \left(\frac{3n-1}{3m}\right),$$

which equation can be more conveniently exhibited as follows

$$\frac{1 \cdot 2 \cdot 4 \cdot 5 \cdot 7 \cdot 8 \cdot 10 \cdots (3m-2)(3m-1)}{3^2 \cdot 6^2 \cdot 9^2 \cdots (3m-3)^2} = \frac{\left(\frac{3}{3m}\right)^2 \left(\frac{6}{3m}\right)^2 \cdots \left(\frac{3n-3}{3m}\right)^2}{\left(\frac{1}{3m}\right) \left(\frac{2}{3m}\right) \left(\frac{4}{3m}\right) \left(\frac{5}{3m}\right) \left(\frac{7}{3m}\right) \cdots \left(\frac{3n-2}{3m}\right) \left(\frac{3n-1}{3m}\right)}.$$

But if in general the common divisor is d and the exponent dn, one will have

$$\left(d \cdot 2d \cdot 3d \cdots (dm-d) \left(\frac{d}{dm}\right) \left(\frac{2d}{dm}\right) \left(\frac{3d}{dm}\right) \cdots \left(\frac{dn-d}{dm}\right)\right)^d$$
$$= 1 \cdot 2 \cdot 3 \cdot 4 \cdots (dm-1) \left(\frac{1}{dm}\right) \left(\frac{2}{dm}\right) \left(\frac{3}{dm}\right) \cdots \left(\frac{dn-1}{dm}\right),$$

which equation can easily be accommodated to any cases, whence the followi-ng theorem deserves it to be noted.

Theorem

§59 *If α was a common divisor of the numbers m and n and the formula $\left(\frac{p}{q}\right)$ denotes the value of the integral $\int \frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}$ extended from $x=0$ to $x=1$, it will be*

$$\left(\alpha \cdot 2\alpha \cdot 3\alpha \cdots (m-\alpha) \left(\frac{\alpha}{m}\right) \left(\frac{2\alpha}{m}\right) \left(\frac{3\alpha}{m}\right) \cdots \left(\frac{n-\alpha}{m}\right)\right)^\alpha$$
$$= 1 \cdot 2 \cdot 3 \cdots (m-1) \left(\frac{1}{m}\right) \left(\frac{2}{m}\right) \left(\frac{3}{m}\right) \cdots \left(\frac{n-1}{m}\right).$$

Proof

The validity of this theorem is already seen from the preceding Scholium; while the common divisor was $= d$ there and the two propounded numbers were dm and dn, here I just wrote m and n instead of them, but instead of their divisor d I wrote the letter α, which kind of divisor the stated equality contains in such a way that one assumes the numbers m and n and hence also $n-\alpha$ and $n-\alpha$ to occur in the continued arithmetic progression α, 2α, 3α etc. In addition, I am forced to confess that this demonstration is of course mainly based on induction and cannot be considered to be rigorous by any means; but because we are nevertheless convicted of its truth, this theorem seems to be worth one's greater attention; there is nevertheless no doubt that a further expansion of integral formulas of this kind will finally lead to a complete proof; but it is an extraordinary specimen of analytical investigation that it was possible for us to see its truth before the complete proof.

COROLLARY 1

§60 So if we substitute the integrals themselves for the signs we introduced, our theorem will be as follows

$$\alpha \cdot 2\alpha \cdot 3\alpha \cdots (m-\alpha) \int \frac{x^{\alpha-1}dx}{\sqrt[n]{(1-x^n)^{n-m}}} \cdot \int \frac{x^{2\alpha-1}dx}{\sqrt[n]{(1-x^n)^{n-m}}} \cdots \int \frac{x^{n-\alpha-1}dx}{\sqrt[n]{(1-x^n)^{n-m}}}$$

$$= \sqrt[\alpha]{1 \cdot 2 \cdot 3 \cdots (m-1) \int \frac{dx}{\sqrt[n]{(1-x^n)^{n-m}}} \cdot \int \frac{xdx}{\sqrt[n]{(1-x^n)^{n-m}}} \cdots \int \frac{x^{n-2}dx}{\sqrt[n]{(1-x^n)^{n-m}}}}.$$

COROLLARY 2

§61 Or if, for the sake of brevity, we set $\sqrt[n]{(1-x^n)^{n-m}} = X$, it will be

$$\alpha \cdot 2\alpha \cdot 3\alpha \cdots (m-\alpha) \int \frac{x^{\alpha-1}dx}{X} \cdot \int \frac{x^{2\alpha-1}dx}{X} \cdots \int \frac{x^{n-\alpha-1}dx}{X}$$
$$= \sqrt[\alpha]{1 \cdot 2 \cdot 3 \cdots (m-1) \int \frac{dx}{X} \cdot \int \frac{xdx}{X} \cdot \int \frac{x^2dx}{X} \cdots \int \frac{x^{n-2}dx}{X}}.$$

General Theorem

§62 *If the divisors of the two numbers m and n are α, β, γ etc. and the formula $\left(\frac{p}{q}\right)$ denotes the value of the integral $\int \frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}$ extended from $x=0$ to $x=1$, the following expressions consisting of integral formulas of this kind will be equal to each other*

$$\begin{aligned}
&\left(\alpha\cdot 2\alpha\cdot 3\alpha\cdots(m-\alpha)\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\cdots\left(\frac{n-\alpha}{m}\right)\right)^{\alpha}\\
=&\left(\beta\cdot 2\beta\cdot 3\beta\cdots(m-\beta)\left(\frac{\beta}{m}\right)\left(\frac{2\beta}{m}\right)\left(\frac{3\beta}{m}\right)\cdots\left(\frac{n-\beta}{m}\right)\right)^{\beta}\\
=&\left(\gamma\cdot 2\gamma\cdot 3\gamma\cdots(m-\gamma)\left(\frac{\gamma}{m}\right)\left(\frac{2\gamma}{m}\right)\left(\frac{3\gamma}{m}\right)\cdots\left(\frac{n-\gamma}{m}\right)\right)^{\gamma}\\
&\text{etc.}
\end{aligned}$$

Proof

The validity of this theorem manifestly follows from the preceding theorem, because every single one of these expressions is equal to this one

$$1\cdot 2\cdot 3\cdots(m-1)\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n-1}{m}\right),$$

which corresponds to the unity as smallest common divisor of the numbers m and n. Therefore, so many expressions of this kind all equal to each other can be exhibited, as there were common divisors of the two numbers m and n.

Corollary 1

§63 Because this formula $\left(\frac{n}{m}\right)$ is $=\frac{1}{m}$ and hence $m\left(\frac{n}{m}\right)=1$, our equal expressions can be more succinctly represented as follows

$$
\begin{aligned}
&\left(\alpha\cdot 2\alpha\cdot 3\alpha\cdots m\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\cdots\left(\frac{n}{m}\right)\right)^{\alpha}\\
=&\left(\beta\cdot 2\beta\cdot 3\beta\cdots m\left(\frac{\beta}{m}\right)\left(\frac{2\beta}{m}\right)\left(\frac{3\beta}{m}\right)\cdots\left(\frac{n}{m}\right)\right)^{\beta}\\
=&\left(\gamma\cdot 2\gamma\cdot 3\gamma\cdots m\left(\frac{\gamma}{m}\right)\left(\frac{2\gamma}{m}\right)\left(\frac{3\gamma}{m}\right)\cdots\left(\frac{n}{m}\right)\right)^{\gamma}.
\end{aligned}
$$

For, even if the number of factors was increased here, the structure of these formulas is nevertheless easily seen.

COROLLARY 2

§64 So if it is $m = 6$ and $n = 12$, because of the common divisors of these numbers, 6, 3, 2, 1, one will have the following four forms all equal to each other

$$
\begin{aligned}
&=\left(6\left(\frac{6}{6}\right)\left(\frac{12}{6}\right)\right)^{6}=\left(3\cdot 6\left(\frac{3}{6}\right)\left(\frac{6}{6}\right)\left(\frac{9}{6}\right)\left(\frac{12}{6}\right)\right)^{3}\\
&=\left(2\cdot 4\cdot 6\left(\frac{2}{6}\right)\left(\frac{4}{6}\right)\left(\frac{6}{6}\right)\left(\frac{8}{6}\right)\left(\frac{10}{6}\right)\left(\frac{12}{6}\right)\right)^{2}\\
&=1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\left(\frac{1}{6}\right)\left(\frac{2}{6}\right)\left(\frac{3}{6}\right)\cdots\left(\frac{12}{6}\right).
\end{aligned}
$$

COROLLARY 3

§65 If the last formula is combined with the penultimate, this equation will arise

$$
\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}=\frac{\left(\frac{2}{6}\right)\left(\frac{4}{6}\right)\left(\frac{6}{6}\right)\left(\frac{8}{6}\right)\left(\frac{10}{6}\right)\left(\frac{12}{6}\right)}{\left(\frac{1}{6}\right)\left(\frac{3}{6}\right)\left(\frac{5}{6}\right)\left(\frac{7}{6}\right)\left(\frac{9}{6}\right)\left(\frac{11}{6}\right)},
$$

but the last compared to the second yields

$$
\frac{1\cdot 2\cdot 4\cdot 5}{3\cdot 3\cdot 6\cdot 6}=\frac{\left(\frac{3}{6}\right)\left(\frac{3}{6}\right)\left(\frac{6}{6}\right)\left(\frac{6}{6}\right)\left(\frac{9}{6}\right)\left(\frac{9}{6}\right)\left(\frac{12}{6}\right)\left(\frac{12}{6}\right)}{\left(\frac{1}{6}\right)\left(\frac{2}{6}\right)\left(\frac{4}{6}\right)\left(\frac{5}{6}\right)\left(\frac{7}{6}\right)\left(\frac{8}{6}\right)\left(\frac{10}{6}\right)\left(\frac{11}{6}\right)}.
$$

SCHOLIUM

§66 Hence infinitely many relations among the integral formulas of the form

$$\int \frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}} = \left(\frac{p}{q}\right)$$

follow, which are even more remarkable, because we were led to them by a completely singular method. And if anyone does not believe that they are true, he or she should consult my observations on these integral formulas[11] and will then hence easily be convinced of their truth for any case. But even if this consideration provides some confirmation, the relations found here are nevertheless of even greater importance, because a certain structure is noticed in them and they are easily generalized to all classes, no matter which number was assumed for the exponent n, whereas in the first treatment the calculation for the higher classes becomes continuously more cumbersome and intricate.

[11] Euler again refers to his paper Öbservationes circa integralia formularum $\int x^{p-1}dx(1-x^n)^{\frac{q}{n}-1}$ posito post integrationem $x=1$". This is paper E321 in the Eneström-Index.

Supplement containing the Proof of the theorem propounded in § 53

It is convenient to derive this proof from the things mentioned above; just take the equation given in §25, which for $f = 1$ having changed the letters is

$$\frac{\int dx\left(\log\frac{1}{x}\right)^{\nu-1}\cdot\int dx\left(\log\frac{1}{x}\right)^{\mu-1}}{\int dx\left(\log\frac{1}{x}\right)^{\nu+\mu-1}} = \varkappa\int\frac{x^{\varkappa\mu-1}dx}{(1-x^{\varkappa})^{1-\nu}},$$

and, using known reductions, represent it in this form

$$\frac{\int dx\left(\log\frac{1}{x}\right)^{\nu}\cdot\int dx\left(\log\frac{1}{x}\right)^{\mu}}{\int dx\left(\log\frac{1}{x}\right)^{\nu+\mu}} = \frac{\varkappa\mu\nu}{\mu+\nu}\int\frac{x^{\varkappa\mu-1}dx}{(1-x^{\varkappa})^{1-\nu}}.$$

Now set $\nu = \frac{m}{n}$ and $\mu = \frac{\lambda}{n}$, but then $\varkappa = n$ so that we have

$$\frac{\int dx\left(\log\frac{1}{x}\right)^{\frac{m}{n}}\cdot\int dx\left(\log\frac{1}{x}\right)^{\frac{\lambda}{n}}}{\int dx\left(\log\frac{1}{x}\right)^{\frac{\lambda+m}{n}}} = \frac{\lambda m}{\lambda+m}\int\frac{x^{\lambda-1}dx}{\sqrt[\varkappa]{(1-x^{n})^{n-m}}},$$

which, for the sake of brevity having used notation introduced above, is more conveniently expressed as follows

$$\frac{\left[\frac{m}{n}\right]\left[\frac{\lambda}{n}\right]}{\left[\frac{\lambda+m}{n}\right]} = \frac{\lambda m}{\lambda+m}\left(\frac{\lambda}{m}\right).$$

Now successively write the numbers 1, 2, 3, 4 . . . n instead of λ and multiply all these equations, whose number is $= n$, and the resulting equation will be

$$\begin{aligned}&\left[\frac{m}{n}\right]^{n}\frac{\left[\frac{1}{n}\right]\left[\frac{2}{n}\right]\left[\frac{3}{n}\right]\cdots\left[\frac{n}{n}\right]}{\left[\frac{m+1}{n}\right]\left[\frac{m+2}{n}\right]\left[\frac{m+3}{n}\right]\cdots\left[\frac{m+n}{n}\right]}\\
&= m^{n}\frac{1}{m+1}\cdot\frac{2}{m+2}\cdot\frac{3}{m+3}\cdots\frac{n}{m+n}\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n}{m}\right)\\
&= m^{n}\frac{1\cdot 2\cdot 3\cdots m}{(n+1)(n+2)(n+3)\cdots(m+n)}\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n}{m}\right).\end{aligned}$$

But in like manner just transform the first part that it is

$$\left[\frac{m}{n}\right]^{n}\frac{\left[\frac{1}{n}\right]\left[\frac{2}{n}\right]\left[\frac{3}{n}\right]\cdots\left[\frac{m}{n}\right]}{\left[\frac{n+1}{n}\right]\left[\frac{n+2}{n}\right]\left[\frac{n+3}{n}\right]\cdots\left[\frac{n+m}{n}\right]},$$

whose agreement with the preceding is reveled by cross multiplication. But because from the nature of these formulas it is

$$\left[\frac{n+1}{n}\right] = \frac{n+1}{n}\left[\frac{1}{n}\right], \quad \left[\frac{n+2}{n}\right] = \frac{n+2}{2}\left[\frac{2}{n}\right], \quad \left[\frac{n+3}{n}\right] = \frac{n+3}{n}\left[\frac{3}{n}\right] \quad \text{etc.,}$$

and since we have m of these formulas here, this first part will become

$$\left[\frac{m}{n}\right]^{n} \frac{n^{m}}{(n+1)(n+2)(n+3)\cdots(n+m)};$$

because this one is equal to the other part exhibited before, namely

$$m^{n}\frac{1\cdot 2\cdot 3\cdots m}{(n+1)(n+2)(n+3)\cdots(n+m)}\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n}{m}\right),$$

we obtain this equation

$$\left[\frac{m}{n}\right]^{n} = \frac{m^{n}}{n^{n}} 1\cdot 2\cdot 3\cdots m\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n}{m}\right),$$

so that

$$\left[\frac{m}{n}\right] = m\sqrt[n]{\frac{1\cdot 2\cdot 3\cdots m}{n^{m}}\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n}{m}\right)};$$

because this equation $\left(\frac{n}{m}\right) = \frac{1}{m}$ completely agrees with the one propounded in §53, its truth is now indeed proved from most certain principles.

Proof of the theorem propounded in §59

Also this theorem needs a more rigorous proof which I will give using the equation established before

$$\frac{\left[\frac{m}{n}\right]\left[\frac{\lambda}{n}\right]}{\left[\frac{\lambda+m}{n}\right]} = \frac{\lambda m}{\lambda+m}\left(\frac{\lambda}{m}\right)$$

as follows. While α is a common divisor of the numbers m and n successively write the numbers α, 2α, 3α etc. up to n instead of λ, whose total amount is $= \frac{n}{\alpha}$, and now multiply all equalities resulting this way that this equation results

$$\left[\frac{m}{n}\right]^{\frac{n}{\alpha}} \frac{\left[\frac{\alpha}{n}\right]\left[\frac{2\alpha}{n}\right]\left[\frac{3\alpha}{n}\right]\cdots\left[\frac{n}{n}\right]}{\left[\frac{m+\alpha}{n}\right]\left[\frac{m+2\alpha}{n}\right]\left[\frac{m+3\alpha}{n}\right]\cdots\left[\frac{m+n}{n}\right]}$$
$$= m^{\frac{n}{\alpha}} \frac{\alpha}{m+\alpha}\cdot\frac{2\alpha}{m+2\alpha}\cdot\frac{3\alpha}{m+3\alpha}\cdots\frac{n}{n+m}\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n}{m}\right).$$

Now transform the left-hand side into this one equal to it

$$\left[\frac{m}{m}\right]^{\frac{n}{\alpha}} \frac{\left[\frac{\alpha}{n}\right]\left[\frac{2\alpha}{n}\right]\left[\frac{3\alpha}{n}\right]\cdots\left[\frac{m}{n}\right]}{\left[\frac{n+\alpha}{n}\right]\left[\frac{n+2\alpha}{n}\right]\left[\frac{n+3\alpha}{n}\right]\cdots\left[\frac{n+m}{n}\right]},$$

which because of $\left[\frac{n+\alpha}{n}\right] = \frac{n+\alpha}{n}\left[\frac{\alpha}{n}\right]$ and so for the remaining ones is reduced to this one

$$\left[\frac{m}{n}\right]^{\frac{n}{\alpha}} \frac{n}{n+\alpha}\cdot\frac{n}{n+2\alpha}\cdot\frac{n}{n+3\alpha}\cdots\frac{n}{n+m},$$

The right-hand side of the equation in like manner is transformed into this one

$$m^{\frac{n}{\alpha}} \frac{\alpha}{n+\alpha}\cdot\frac{2\alpha}{n+2\alpha}\cdot\frac{3\alpha}{n+3\alpha}\cdots\frac{m}{n+m}\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\cdots\left(\frac{n}{m}\right),$$

whence this equation results

$$\left[\frac{m}{n}\right]^{\frac{n}{\alpha}} n^{\frac{m}{\alpha}} = n^{\frac{n}{\alpha}}\alpha\cdot 2\alpha\cdot 3\alpha\cdots m\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\cdots\left(\frac{n}{m}\right)$$

and hence

$$\left[\frac{m}{n}\right] = m\sqrt[n]{\frac{1}{n^m}\left(\alpha\cdot 2\alpha\cdot 3\alpha\cdots m\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\cdots\left(\frac{n}{m}\right)\right)^{\alpha}}$$

which equation compared to the preceding yields this equation

$$\left(\alpha\cdot 2\alpha\cdot 3\alpha\cdots m\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\cdots\left(\frac{n}{m}\right)\right)^{\alpha} = 1\cdot 2\cdot 3\cdots m\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n}{m}\right),$$

what is to be understood for all common divisors of the two numbers m and n.